J. of Basic Sciences, Cheju Nat'l Univ.
15(2), 123~132, 2002

기초과학연구 제주대학교
15(2), 123~132, 2002

Zero-term Rank Preservers of Fuzzy Matrices

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Abstract

Zero-term rank of a matrix is the minimum number of lines(rows or columns) needed to cover all the zero entries of the given matrix. In this thesis, we characterize the zero-term rank preservers of the $m \times n$ matrices over a fuzzy semiring.

Keywords: Fuzzy matrix; Zero-term rank; Term rank; (P, Q, B)-operator; Zero-term rank preserver

AMS Subject Classifications: 08A72, 15A03, 15A04

1 Introduction

There is much literature on the study of those linear operators on matrices that leave certain properties or subsets invariant. Boolean matrices also have been the subject of research by many authors. Beasley and Pullman characterized those linear operators that preserve Boolean rank in [1] and term rank of matrices over semirings in [4].

Beasley, Song and Lee obtained characterization of linear operators that preserve zero-term rank of Boolean matrices in [3].

In this thesis, we consider the zero-term rank of fuzzy matrices. We obtain characterizations of those linear operators that preserve zero-term rank of $m \times n$ matrices over a fuzzy semiring $\mathbf{F}$.

2 Definitions and Notations

A *semiring* ([4]) consists of a set **S**, and two binary operations on **S**, addition(+) and multiplication(·), such that

(1) $(\mathbf{S}, +)$ is an Abelian monoid under addition (identity denoted by 0);
(2) $(\mathbf{S}, \cdot)$ is a monoid under multiplication (identity denoted by 1);
(3) multiplication distributes over addition ;
(4) s0 = 0s =0 for all s $\in$ **S** ; and
(5) $0 \neq 1$.

A *fuzzy semiring* consists of a set $\mathbb{F} = \{\beta \mid 0 \leq \beta \leq 1\ ,\ \beta \in \mathbb{R}\}$ and two binary operations on $\mathbb{F}$, addition and multiplication. The operations are defined as follows:

$$x + y = max(x, y) \qquad \text{and} \qquad xy = min(x, y).$$

Usually $\mathbb{F}$ denotes both the fuzzy semiring and the set. Let $M_{m,n}(\mathbb{F})$ denote the set of all $m \times n$ matrices with entries in a fuzzy semiring $\mathbb{F}$. We call a matrix in $M_{m,n}(\mathbb{F})$ as a *fuzzy matrix*. The zero matrix and the $n \times n$ identity matrix I_n are defined as if $\mathbb{F}$ were a field. Addition, multiplication by scalar, and the product of matrices are also defined as if $\mathbb{F}$ were a field. The $m \times n$ matrix of $1's$ is denoted $J_{m,n}$. The $m \times n$ matrix of whose entries are zero except its (i, j)th, which is 1, is denoted E_{ij}. We call E_{ij} a *cell*. The set of all cells is denoted $\Delta = \{E_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ and the set of its indices is denoted $\mathcal{E} = \{(i, j) : 1 \leq i \leq m, 1 \leq j \leq n\}$. The set of all cells spans $M_{m,n}(\mathbb{F})$.

If A and B are in $M_{m,n}(\mathbb{F})$, we say A *dominates* B (written $A \geq B$ or $B \leq A$) if $a_{ij} \geq b_{ij}$ for all i, j. This provides a reflexive, transitive relation on $M_{m,n}(\mathbb{F})$.

The *zero-term rank* of a matrix X, $z(X)$, is the minimum number of lines(rows or columns) needed to cover all the zero entries in X. And the *term rank* of a matrix X, $t(X)$, is the minimum number of lines(rows or columns) needed to cover all the nonzero entries in X.

LEMMA 2.1. *For $A, B \in M_{m,n}(\mathbb{F})$, $A \geq B$ implies that $z(A) \leq z(B)$.*

Proof. If $z(B) = k$, then there are k lines which cover all zero entries in B. Since $A \geq B$, this k lines can also cover all zero entries in A. Hence $z(A) \leq k = z(B)$. □

3 Linear Operators That Preserve Zero-term Rank Of Fuzzy Matrices

A function T mapping $M_{m,n}(\mathbf{F})$ into itself is called an *operator* on $M_{m,n}(\mathbf{F})$. The operator T is *linear* if $T(\alpha A + \beta B) = \alpha T(A) + \beta T(B)$ for all $\alpha, \beta \in \mathbf{F}$ and all $A, B \in M_{m,n}(\mathbf{F})$.

DEFINITION 3.1. Let T be a linear operator on $M_{m,n}(\mathbf{F})$. If $z(T(X)) = k$ whenever $z(X) = k$ for all X in $M_{m,n}(\mathbf{F})$, we say T *preserves zero-term rank* k. If T preserves zero-term rank k for every $k \leq min\{m, n\}$, then we say T *preserves zero-term rank.*

DEFINITION 3.2. Let T be a linear operator on $M_{m,n}(\mathbf{F})$. If $t(T(X)) = k$ whenever $t(X) = k$ for all X in $M_{m,n}(\mathbf{F})$, we say T *preserves term rank* k. If T preserves term rank k for every $k \leq min\{m, n\}$, then we say T *preserves term rank.*

Which linear operators on $M_{m,n}(\mathbf{F})$ preserve zero-term rank? The operations of permuting rows, permuting columns, and (if $m = n$) transposing the matrices in $M_{m,n}(\mathbf{F})$ are all linear operators that preserve zero-term rank of the matrices on $M_{m,n}(\mathbf{F})$. If we take a fixed $m \times n$ matrix B in $M_{m,n}(\mathbf{F})$, then its *Schur product* is defined $B \circ X = [b_{ij}x_{ij}]$ for all X in $M_{m,n}(\mathbf{F})$.

LEMMA 3.3. *Suppose that T is an operator on $M_{m,n}(\mathbf{F})$ such that $T(X) = B \circ X$, where B is fixed in $M_{m,n}(\mathbf{F})$, none of whose entries is zero in $\mathbf{F}$. Then T is a linear operator which preserves zero-term rank.*

Proof. For all α, $\beta \in \mathbf{F}$ and A, $B \in M_{m,n}(\mathbf{F})$, we have the following equality;

$$\begin{aligned} T(\alpha X + \beta Y) &= B \circ (\alpha X + \beta Y) = B \circ (\alpha X) + B \circ (\beta Y) \\ &= \alpha(B \circ X) + \beta(B \circ Y) = \alpha T(X) + \beta T(Y). \end{aligned}$$

Hence T is linear. The fact T preserves zero-term rank follows the definition of Schur product. □

That these operations and their compositions are the only zero-term rank preservers is one of the consequences of Theorem 3.10 below. Such operators are described more formally in the following definition.

DEFINITION 3.4. If P and Q are $m \times m$ and $n \times n$ permutation matrices, respectively and B is an $m \times n$ matrix, none of whose entries is zero in $\mathbf{F}$, then T is a (P, Q, B)-*operator* if

(1) $T(X) = P(B \circ X)Q$ for all X in $M_{m,n}(\mathbf{F})$ or
(2) $m = n$, and $T(X) = P(B \circ X^t)Q$ for all X in $M_{m,n}(\mathbf{F})$.

THEOREM 3.5. ([4]) *If* $\mathbf{S}$ *is any semiring, then the following are equivalent for any linear operator* T *on* $M_{m,n}(\mathbf{S})$*;*

(1) T *is a* (P, Q, B)*-operator;*
(2) T *preserves term rank;*
(3) T *preserves term ranks 1 and 2.*

We will show that the zero-term rank preservers are also of the same form as the term rank preservers. For this purpose, we define $T' : \mathcal{E} \to \mathcal{E}$ by $T'(i,j) = (u,v)$ whenever $T(E_{ij}) = b_{ij}E_{uv}$ with $0 < b_{ij} \leq 1$, where $\mathcal{E} = \{(i,j) : 1 \leq i \leq m, 1 \leq j \leq n\}$ is the set of all indices.

LEMMA 3.6. *Suppose that* T *preserves zero-term ranks* 0 *and* 1*. Then* T *maps a cell onto a cell with a scalar multiple and hence* T' *is a bijection on the set* $\mathcal{E}$*.*

Proof. If $T(E_{ij}) = 0$ for some $E_{ij} \in \Delta$, then we can choose $mn - 1$ cells $E_1, E_2, \cdots, E_{mn-1}$ which are different from E_{ij} such that

$$\begin{aligned} T(J) &= T(E_{ij} + \textstyle\sum_{h=1}^{mn-1} E_h) = T(E_{ij}) + T(\sum_{h=1}^{mn-1} E_h) \\ &= 0 + T(\textstyle\sum_{h=1}^{mn-1} E_h) = T(\sum_{h=1}^{mn-1} E_h). \end{aligned}$$

But $z(J) = 0$ and $z(\sum_{h=1}^{mn-1} E_h) = 1$. Since T preserves zero-term ranks 0 and 1, we have $z(T(J)) = 0$ and $z(T(\sum_{h=1}^{mn-1} E_h)) = 1$. This is a contradiction because $0 = z(T(J)) = z(T(\sum_{h=1}^{mn-1} E_h)) = 1$. Hence $T(E_{ij})$ dominates at least one cell with a scalar multiple. That is, $T(E_{ij}) \geq b_{ij}E_{uv}$ for some $E_{uv} \in \Delta$ with $0 < b_{ij} \leq 1$.

For some cell $E_{ij} \in \Delta$, suppose $T(E_{ij}) \geq b_{ij}E_{kl} + b'_{ij}E_{uv}$ for some $E_{kl}, E_{uv} \in \Delta$ with $0 < b_{ij}, b'_{ij} \leq 1$. If E_{rs} is a cell different from E_{kl} and E_{uv}, then we can choose one cell E_h such that $T(E_h)$ dominates $b_h E_{rs}$ because T preserves zero-term rank 0. Since the number of cells except for both E_{kl} and E_{uv} is $mn - 2$, there exist at most $mn - 1$ cells $E_1, E_2, \cdots, E_{mn-1}$ containing E_{ij} such that

$$z(T(\sum_{h=1}^{mn-1} E_h)) = 0.$$

But $z(\sum_{h=1}^{mn-1} E_h)) = 1$. This contradicts that T preserves zero-term rank 1. Hence $T(E_{ij}) = b_{ij}E_{uv}$ for all $E_{ij} \in \Delta$. That is, T maps a cell into a cell with a scalar multiple.

Now we show that T' is a bijection on $\mathcal{E}$. If $T'(i,j) = T'(r,s) = (u,v)$ for some different indices (i,j) and (r,s), then we have

$$\begin{aligned} T(J) &= T(\{J - (E_{ij} + E_{rs})\} + (E_{ij} + E_{rs})) \\ &= T(J - (E_{ij} + E_{rs})) + T(E_{ij} + E_{rs}) \\ &= T(J - (E_{ij} + E_{rs})) + T(E_{ij}) + T(E_{rs}). \end{aligned}$$

Since $T'(i,j) = T'(r,s) = (u,v)$, we have $T(E_{ij}) = b_{ij}E_{uv}$ and $T(E_{rs}) = b_{rs}E_{uv}$ with $0 < b_{ij}, b_{rs} \leq 1$. Without loss of generality, we may assume that $b_{ij} \geq b_{rs}$. Then the above equality implies that we have

$$\begin{aligned} T(J) &= T(J - (E_{ij} + E_{rs})) + T(E_{ij}) + T(E_{rs}) \\ &= T(J - (E_{ij} + E_{rs})) + b_{ij}E_{uv} + b_{rs}E_{uv} \\ &= T(J - (E_{ij} + E_{rs})) + b_{ij}E_{uv} \\ &= T(J - (E_{ij} + E_{rs})) + T(E_{ij}) \\ &= T(J - (E_{ij} + E_{rs}) + E_{ij}) \\ &= T(J - E_{rs}). \end{aligned}$$

But $z(J-E_{rs})=1$ and $z(J)=0$. This contradicts that T preserves zero-term ranks 0 and 1. Therefore T' is an injection on $\mathcal{E}$ and hence T' is a bijection on $\mathcal{E}$. □

LEMMA 3.7. *If T preserves zero-term ranks 0 and 1, then T preserves term rank 1.*

Proof. Suppose that T does not preserve term rank 1. Then there exist some cells E_{ij} and E_{il} on the same row(or column) such that $T(E_{ij}+E_{il})=T(E_{ij})+T(E_{il})=b_{ij}E_{pq}+b_{il}E_{rs}$ with $p\neq r$ and $q\neq s$, where $T'(i,j)=(p,q)$ and $T'(i,l)=(r,s)$. Since T preserves zero-term ranks 0 and 1, we have that T' is bijective on $\mathcal{E}$ by Lemma 3.6. Hence we have $T(J)=B=(b_{uv})_{m\times n}$ for some $B\in M_{m,n}(\mathbf{F})$ and $0<b_{uv}\leq 1$. Since T preserves zero-term rank 1 and $z(J-E_{ij}-E_{il})=1$, we have $z(T(J-E_{ij}-E_{il}))=1$. But the image of $J-E_{ij}-E_{il}$ has zeros in (p,q) and (r,s) positions because $T(E_{ij}+E_{il})=T(E_{ij})+T(E_{il})=b_{ij}E_{pq}+b_{il}E_{rs}$. Thus we have $z(T(J-E_{ij}-E_{il}))=2$. This is a contradiction. Hence T preserves term rank 1. □

LEMMA 3.8. *If T preserves zero-term ranks 0 and 1, then T maps a row of a matrix onto a row(or column if $m=n$) with scalar multiple in $\mathbf{F}$.*

Proof. Suppose that T does not map a row into a row with scalar multiple (or column if $m=n$). Then T does not preserve term rank 1. This contradicts to Lemma 3.7. Hence T maps a row into a row with scalar multiple (or column if $m=n$). Since T preserves zero-term ranks 0 and 1, we have that T' is bijective on $\mathcal{E}$ by Lemma 3.6. Then the bijectivity of T' implies that T maps a row onto a row with scalar multiple (or may be a column if $m=n$). □

LEMMA 3.9. *For the case $m=n$, suppose that T preserves zero-term ranks 0 and 1. If T maps a row onto a row(or column) with scalar multiples in $\mathbf{F}$, then all rows of a matrix must be mapped some rows(or columns, respectively) with scalar multiples in $\mathbf{F}$.*

Proof. Since T preserves zero-term ranks 0 and 1, T' is bijective on $\mathcal{E}$ by Lemma

3.6. Let $R_i = \sum_{j=1}^{n} E_{ij}$ and $C^j = \sum_{i=1}^{n} E_{ij}$, where $i, j = 1, 2, \cdots, n$. Suppose T maps a row, say R_1, onto an ith row R_i with scalar multiple B_i and another row, say R_2, onto a jth column C^j with scalar multiple B^j. That is, $T(R_1) = B_i \circ R_i$ and $T(R_2) = B^j \circ C^j$. Then $R_1 + R_2$ has $2n$ cells but $B_i \circ R_i + B^j \circ C^j$ has $2n - 1$ cells. This contradicts to the bijectivity of T' on $\mathcal{E}$. Hence all rows must be mapped some rows(or columns, respectively) with scalar multiple. □

We have the following characterization theorem for zero-term rank preserver on $M_{m,n}(\mathbf{F})$.

THEOREM 3.10. *Suppose that T is a linear operator on $M_{m,n}(\mathbf{F})$. Then the following statements are equivalent;*

(1) *T is a (P, Q, B)-operator;*

(2) *T preserves zero-term rank;*

(3) *T preserves zero-term ranks 0 and 1.*

Proof. (1) $\Rightarrow$ (2): Suppose that T is a (P, Q, B)-operator and the zero-term rank of X is k, that is , $z(X) = k$. Since T is a (P, Q, B)-operator, we have $T(X) = P(B \circ X)Q$ or $m = n$, and $T(X) = P(B \circ X^t)Q$, where P and Q are permutation matrices and B is an $m \times n$ matrix over $\mathbf{F}$, none of whose entries is zero. Hence $z(T(X)) = z(P(B \circ X)Q) = k = z(X)$ or $z(T(X)) = z(P(B \circ X^t)Q) = k = z(X)$. Since k is an arbitrary, we have that T preserves zero-term rank. (2) $\Rightarrow$ (3): It is clear. (3) $\Rightarrow$ (1): Suppose that T preserves zero-term ranks 0 and 1. Then T' is a bijection on $\mathcal{E}$ by Lemma 3.6. Lemmas 3.8 and 3.9 imply that T maps all rows of a matrix onto rows with scalar multiples or columns onto columns with scalar multiples. Thus, for all $m \times n$ matrix X in $M_{m,n}(\mathbf{F})$, $T(X) = P(B \circ X)Q$ or $m = n$, and $T(X) = P(B \circ X^t)Q$ with some permutation matrices P and Q and B is a fixed $m \times n$ matrix over $\mathbf{F}$, none of whose entries is zero. Hence T is a (P, Q, B)-operator.
□

LEMMA 3.11. *For A, B in $M_{m,n}(\mathbf{F})$, $A \geq B$ implies $T(A) \geq T(B)$.*

Proof. By definition of $A \geq B$, we have $a_{ij} \geq b_{ij}$ for all i, j. Using $A = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}E_{ij}$ and $B = \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}E_{ij}$, we have

$$\begin{aligned} T(A) &= T(\textstyle\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}E_{ij}) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}T(E_{ij}) \\ &\geq \textstyle\sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}T(E_{ij}) = T(\sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}E_{ij}) = T(B) \end{aligned}$$

because of linearity and $a_{ij} \geq b_{ij}$. Hence $T(A) \geq T(B)$. □

We say that a linear operator T *strongly preserves zero-term rank* k provided that $z(T(A)) = k$ if and only if $z(A) = k$.

LEMMA 3.12. *If T strongly preserves zero-term rank* 1, *then T preserves zero-term rank* 0.

Proof. Suppose that T strongly preserves zero-term rank 1. Since $z(J) \neq 1$, we have $z(T(J)) = 0$ or $z(T(J)) \geq 2$. Suppose $z(T(J)) \geq 2$. Let A be any matrix in $M_{m,n}(\mathbf{F})$. Then $J \geq A$ and so $T(J) \geq T(A)$ by Lemma 3.11. Lemma 2.1 implies $2 \leq z(T(J)) \leq z(T(A))$. Hence $z(T(A)) \geq 2$ for all $A \in M_{m,n}(\mathbf{F})$. For any cell $E_{ij} \in \Delta$, let $A = J - E_{ij}$. Then we have $z(A) = z(T(J - E_{ij})) = 1$. Since T strongly preserves zero-term rank 1, $z(T(A)) = z(T(J - E_{ij})) = 1$. This is impossible. Hence $z(T(J)) = 0$. This means that T preserves zero-term rank 0. □

THEOREM 3.13. *Suppose T is a linear operator on $M_{m,n}(\mathbf{F})$. Then T preserves zero-term rank if and only if it strongly preserves zero-term rank* 1.

Proof. Suppose that T strongly preserves zero-term rank 1. Then Lemma 3.12 implies that T preserves zero-term rank 0. By Theorem 3.10, T preserves zero-term rank. Conversely, suppose that T preserves zero-term rank. If $z(T(X)) = 1$ and $z(X) \neq 1$, then $z(X) = 0$ or $z(X) \geq 2$. If $z(X) = 0$(or $z(X) \geq 2$), then $z(T(X)) = 0$(or $z(X) \geq 2$) by hypothesis. This contradicts to $z(T(X)) = 1$. Hence T strongly preserves zero-term rank 1. □

Thus we obtained the characterizations of linear operators that preserves zero-term rank of fuzzy matrices. It turns out that the linear operator is a (P, Q, B)-

operator, which equals term rank preserver. Also, we obtained several kinds of conditions that are equivalent to a (P, Q, B)-operator.

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퍼지행렬의 영향계수 보존자

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요 약

행렬의 영향 계수는 그 행렬에 나타나는 모든 영 원소들을 덮을 수 있는 행과 열의 극소수로 정의된다. 본 논문에서는 퍼지집합에서 원소를 갖는 퍼지행렬들을 생각한다. 이 퍼지행렬들의 영향계수 보존자를 연구하여 그 형태를 규명하였고, 또 이 보존자와 필요충분조건들을 찾아서 그 동치성을 증명하였다. 곧, 모든 영향계수를 보존하는 선형연산자의 형태는 주어진 행렬의 좌우측에 순환행렬을 곱하며, 주어진 행렬의 영 아닌 원소들을 영 되게 하지 않는 행렬의 Schur 곱의 형태로 나타남을 밝혔다.