Introduction to Common Knowledge

Kim, Jin Ock*

CONTENTS

Ⅰ. Introduction

Knowledge and Interactive knowledge are key elements in Economic theory. For example in Game theory, the rationality of agents in the game is a common knowledge among agents. Loosely speaking, The common knowledge is that players know that their opponents know that they know ad infinitum. The idea of common knowledge is a useful tool for understanding the information structure on which the equilibrium depends.

In chapter 1, we review the basic measure theoretic concept by following the standard text book on probability theory[1]. In chapter 2, Given the knowledge of measure theory, we define the knowledge function and examine several properties of knowledge function. Equipped with this, we demonstrate Aumann(1976)'s famous result of "Impossibility of agreeing to disagree." Lastly, we demonstrate the paper "Common Knowledge of an Aggregate of expectation"(Niellson, Brandenburger, Geanakoplos,

* Professor, Dept. of Economics, Cheju National University
1) There are many text books on measure theory such as Grimmet and Stirzaker(1992)

McKelvey and Page(1990). This paper says that public observation of an aggregate of individual expectation such as prices leads to consensus.

II. Algebras and $\sigma-algebra$

2.1 Algebras

Definition 1: let X be a space. A collection $\mathcal{A}$ of subsets of X is an algebra on X, if and only if it satisfies all the following properties:

1. Both X and null set belong to $\mathcal{A}$.
2. For all $A \in \mathcal{A}$, $A^c \in \mathcal{A}$.
3. For all $A, B \in \mathcal{A}$, $A \cup B \in \mathcal{A}$.

Proposition 1 : Let X be a space, $\mathcal{A}$ an algebra on X, and A, B two subsets of X that belong to $\mathcal{A}$. Then,

1. $A \cap B \in \mathcal{A}$.
2. $A \setminus B \in \mathcal{A}$.
3. $A \Delta B \in \mathcal{A}$.

Proof :

1. $A \cap B = (A^c \cup B^c)^c \in \mathcal{A}$ since A, B $\in \mathcal{A}$ and $\mathcal{A}$ is an algebra.
2. $A \setminus B = A \cap B^c \in \mathcal{A}$ by the previous item.
3. $A \Delta B = (A \setminus B) \cup (B \setminus A)$ by the previous item.

It is obvious that any finite union of elements of algebra is an element of an algebra. The same story holds for any finite intersection of elements of an algebra.

Literally saying, the collection $\mathcal{A}$ of subsets of X is just the set. In this case each element of $\mathcal{A}$ is some subset of X. Given a space X, the set consisting of subsets of X as elements of it could be algebra if it satisfies above properties given in definition 1. Let's consider some collection $\mathcal{A}$ = {X, null set}. The collection $\mathcal{A}$ is an algebra because it satisfies the properties of algebra. Let X = {1, 2, 3, 4}. The collection A = {{1,2,3,4},{null set},{1},{2.3,4}}is an algebra because: 1. the whole set X and null set belong to $\mathcal{A}$, 2. for any element in A, the complement of A is in the collection A, and 3. any union of two elements in the collection A belongs to the collection A. Obviously the power set of all subsets of X is an algebra.

Proposition 2 : Let X be a space, and let $(\mathcal{A}_\alpha)_{\alpha \in I}$ be a non-empty collection of algebras on X. Then the collection $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$ is an algebra.

Proof :

1. Both X and null set belong to any algebra concerned, so they belong to the intersection of algebras.
2. Suppose A belongs to the intersection of algebra. Then A belong to any algebra concerned. This means that the complement of A belongs to any algebra by the property of algebra. Therefore the complement of A belongs to the intersection of algebra.
3. Suppose A and B belong to the intersection of algebras concerned. Then A and B belong to any algebra concerned. Therefore A union B belong to any algebra, which implies that A union B is an element of the intersection of algebras.

Proposition 3 : Let X be a space and let M be a collection of subsets of X. Let F be the collection of all the algebras on X that contains M. Then,

1. F is not a null set
2. The intersection of all the elements of F is an algebra that contains M.

Proof :

1. The power set of all subsets of X belongs to F since it is an algebra containing M. So F is not

a null set.

2. The intersection of all the algebras in F is an algebra by Proposition 2. Since each one of them contains M, so does the intersection.

Definition 2 : Let X be a space, and let M be a collection of subsets of X. The intersection of all the algebras on X that contains M is called the algebra generated by M, it is denoted by $\mathcal{A}$(M).

Above Definition 2 says that $\mathcal{A}$(M) is the smallest algebra among the algebras that contains M.

Proposition 4 : Let X be a space. M a collection of subsets of X, and $\mathcal{A}$ an algebra that contains M. Then $\mathcal{A}$=$\mathcal{A}$(M) if and only if $\mathcal{A}$ is contained in every algebra that contains M.

Proof : Suppose $\mathcal{A}$=$\mathcal{A}$(M). Definition 2 says that $\mathcal{A}$(M) is contained in every algebra that contains M. So does $\mathcal{A}$. Conversely suppose that $\mathcal{A}$ is contained in any algebra containing M. So it is obvious that $\mathcal{A}\subseteq\mathcal{A}$(M). By Definition of $\mathcal{A}$(M), $\mathcal{A}(M)\subseteq\mathcal{A}$.

Proposition 5 : Let M and K be two collections of subsets of a space of X. If $M\subseteq K\subseteq\mathcal{A}(M)$, then $\mathcal{A}(M)=\mathcal{A}(K)$.

Proof : $\mathcal{A}(K)\subseteq\mathcal{A}(M)$ since $\mathcal{A}$(M) is an algebra that contains K and $\mathcal{A}$(K) is the smallest algebra containing K by definition of $\mathcal{A}$(K). $\mathcal{A}(M)\subseteq\mathcal{A}(K)$ since $\mathcal{A}$(K) contains M and $\mathcal{A}$(M) is the smallest algebra containing M.

Above proposition states that if $\mathcal{A}$(M) contains K a collection of subsets of X bigger than M, then $\mathcal{A}$(K) the smallest algebra generated by K is just the algebra $\mathcal{A}$(M).

2.2 $\sigma-Algebras$

We are now stating the stronger concept than the algebra.

Definition 3 : Let X be a space. A collection $\mathcal{A}$ of subset of X is a $\sigma - a\lg era$ on X , if and only if it satisfies all the following properties:

1. Both X and null set belong to $\mathcal{A}$
2. For all $A \in \mathcal{A}$, the complement of A belongs to $\mathcal{A}$
3. For all sequences $(A_n)_{n=1}^{\infty}$ of $\mathcal{A}$, $\cup_{n=1}^{\infty} A_n$ belongs to $\mathcal{A}$.

It should be clear that every sigma algebra is an algebra, since every union of two sets A and B can be written as a countable union $(A \cup B \cup \phi \cup \phi ...)$. Consider the following example of showing that not all algebras are sigma algebras. Let X = {1, 2, 3, ⋯}, and let $\mathcal{A}$= $\{A \subseteq X : A$ is finite or the complement of A is finite}. It is easy to show that $\mathcal{A}$ is an algebra. Both X and null set are in $\mathcal{A}$ since null set is finite. Let A, B $\in \mathcal{A}$. If both A and B are finite, then so is $A \cup B$. Otherwise at least one of them is infinite, but its complement is finite. Let such a set be A. Then the complement of $A \cup B$ is a subset of the complement of A which is finite. So the complement of $A \cup B$ is finite and therefore is in $\mathcal{A}$. The complement of A is in $\mathcal{A}$ because the complement of it is A and A is in $\mathcal{A}$. On the other hand $\mathcal{A}$ is not a sigma algebra. Indeed each even number is in $\mathcal{A}$ but the union of all of the even numbers is not in $\mathcal{A}$.

All of the previous results for algebra hold for sigma algebra if we substitute sigma algebra for algebra in the statements concerning algebra. For example: 1. the intersection of sigma algebra is sigma algebra, 2 The intersection of all the sigma algebras on X that contains some collection M is called the smallest sigma algebra generated by M.

Ⅲ. The impossibility of Agreeing to Disagree

Given the previous knowledge on sigma algebra, we are in a position to demonstrate Aumann(1976)'s famous result of "Impossibility of agreeing to disagree." First we introduce information partition for each agent.

Let I be a finite set of agents. Let (Ω, Σ, μ) be a probability space where 1. Ω is a countable set of states. 2. Σ is a sigma algebra of events. 3. μ is probability measure on Σ.

For each $i \in I$, Π_i is an information partition of Ω into measurable sets of positive measure. The elements of Π_i are disjoint. For $\omega \in \Omega$, $\Pi_i(w)$ denotes the element of the partition Π_i that contains w. For $w, w' \in \Pi_i(w)$, $\Pi_i(w) = \Pi_i(w')$. To have a clear understanding, consider following example. Suppose $\Omega = \{1,2,3,4,5,6,7,8,9\}$; there are nine possible states in the world. And let the information partition for agent A be $\Pi_A = \{\{1,2,3\}, \{4,5\}, \{6,7,8\}, \{9\}\}$. For $w = 4, w' = 5, \Pi_i(w = 4) = \Pi_i(w' = 5) = \{4,5\}$. We denote by F_i the sigma algebra generated by the information partition Π_i.

Proposition 1 : F_i is the set of all unions of elements of Π_i.

Proof : Denote by M the set of all unions of elements of Π_i. First we need to show that M is a subset of F_i. Note that F_i is the intersection of all the sigma algebras that contains Π_i. But any sigma algebra containing Π_i must contain all the countable union of elements of Π_i. So M is a subset of F_i. Next we show that F_i is a subset of M. It is enough to show that M is itself a sigma algebra containing Π_i. To see this, First note that Ω belongs to M since it is the union of all the elements of Π_i. Also null set is the element of M since it is the empty union. Second, if A,B $\in$ M, union of A and B belongs to M since it is the union of elements of Π_i. Lastly, let A $\in$ M, then the complement of A consists of elements of Π_i, not the element of A. This implies that the complement of A is some union of elements of Π_i.

Above proposition says that given the information partition for each agent, we can form the sigma algebra by simply considering the set of all unions of elements of the information partition.

Definition 1 : We say that agent i knows event at the state w if $\Pi_i(w) \subseteq$ E.

How do we interpret this? To interpret this, let Ω and Π_A be given as above. And suppose the information partition for the agent B $\Pi_B = \{\{1,2\},\{3,4,5\},\{6,7\},\{7,8,9\}\}$. Consider the event $E = \{1,2,3,4\}$. Suppose that the true state is 2. In this case agent A knows that the true state is either 1, 2, or 3: Agent A knows event E since event {1, 2, 3} is a subset of E. Does agent B know event E? Yes. Given the true state is 2, Agent B knows that the true state is 1 or 2. So agent B knows the event E since {1,2} is a subset of E. Consider the event F={2,3,4,5}. Does agent A know the event F? He does not know the event F since {1,2,3} is not a subset of F. The same story holds with agent B. Does A know that B knows E? No! Given the true state w=2, agent A knows that the true state is in {1, 2, 3}. In this case he knows that either (1) agent B knows that the true state is {1, 2}; or (2) agent B knows the true state is in {3, 4, 5}. In case (1), agent B does know E, but in case (2) agent B does not know E.

Definition 2 : Let $K_i(E)$ be the event " i knows E." That is, $K_i(E) = \{w \in \Omega : \Pi_i(w) \subseteq E\}$.

Proposition 2 : Agent i knows event E at w if and only if there exists $A \in F_i$ such that $w \in A \subseteq E$

Proo f: $w \in K_i(E)$ implies that there exists $\Pi_i(w)$ which is a subset of E. $\Pi_i(w)$ is the set A we are looking for since $w \in \Pi_i(w) \subseteq F_i$. Conversely assume that $w \in A \subseteq E$ for some $A \in F_i$. A is a union of elements of Π_i since $A \in F_i$. Since $w \in A$, we must have $\Pi_i(w) \subseteq A$. Therefore $\Pi_i(w) \subseteq$ E, and $w \in K_i(E)$.

Proposition 3 : Let E be an event, and let M={ $A \in F_i$: $A \subseteq E$ } be the set of F_i measurable subsets of E. Then $K_i(E) = \cup_{A \in M} A$.

Proof : Let $w \in K_i(E)$. Then by proposition 2, there exists $A \in F_i$ such that $w \in A \subseteq E$. This means that $w \in A \in M$ and consequently $w \in \cup_{A \in M} A$. Conversely assume $w \in \cup_{A \in M} A$. Then there is $A \in F_i$ such that $w \in A \subseteq E$. By proposition 2, $w \in K_i(E)$.

Above proposition 3 implies that $K_i(E)$ is an element of F_i and a subset of E.

Proposition 4 : For all events $E \in \Sigma$, $K_i(E)$=E if and only if $E \in F_i$.

Proof: Suppose $K_i(E)$=E. $K_i(E)$ belongs to F_i. So does E. Conversely if $E \in F_i$, Then E belongs to **M** since E is a subset of itself and belongs to F_i. By proposition 3 $E \subseteq K_i(E)$. And we know that $K_i(E)$ belongs to F_i and a subset of E. $K_i(E)$=E.

Proposition5 : For all sequences of events $(E_n)_{n\geq 1}, K_i(\cap_{n\geq 1} E_n) = \cap_{n\geq 1} K_i(E_n)$.

Using the proposition 3, we can prove above claim 5. It is also obvious that $K_i(E)$ is a subset of $K_i(F)$ if E and F are measurable and E is a subset of F.

Until now we have examined several properties of knowledge function given the sigma algebra generated by the information partition. We extend Definition 1 of the knowledge of each agent to have the Definition of common knowledge.

Definition 3 : An event of E is common knowledge at $w \in \Omega$ if there exists an event $A \in F_i$ for all $i \in I$ such that $w \in A \subseteq E$.

Suppose there are two agents A and B in the economy. Then we can say an event E is common knowledge at some true state if player A knows E, player B knows E , player A knows that B knows E, player B knows that A knows E, and so on indefinitely. This loose statement for common knowledge can be rigorously stated in mathematical terms. Following Monderer and Samet (1989), given an event E, we define the following events recursively:

$$E^0 = E$$

$$E^n = \cap_{i \in N} K_i(E^{n-1})$$

And $C(E) = \cap_{n \geq 1} E^n$.

Proposition 6(Monder and Samet (1989)) : E is common knowledge at w if and only if $w \in C(E)$.

Proof : Assume that E is common knowledge at w. Then by definition of common knowledge, there exists an event $A \in F_i$ for all $i \in I$ such that $w \in A \subseteq E$. It is enough to show that $A \subseteq C(E)$. Propositions 3 and 4 say that $A = K_i(A) \subseteq K_i(E) \quad \forall i \in N$, whish implies $A \subseteq \cap_{i \in N} K_i(E) = E^1$. Assume that for $n \geq 1, A \subseteq E^n$. Then again by propositions 3 and 4, $A = K_i(A) \subseteq K_i(E^n) \quad \forall i \in N$ which implies $A \subseteq \cap_{i \in N} K_i(E^n) = E^{n+1}$. We have proved inductively that $A \subseteq E^n$ for all $n \geq 1$. Therefore, $A \subseteq \cap_{n \geq 1} E^n = C(E)$. Conversely assume now that $w \in C(E)$. We need to show that there is a common event $A \in F_i$ for all i such that $w \in A \subseteq E$. We will show that $C(E)$ is such an event we are looking for. Firstly we know $C(E) \subseteq E$ since $C(E) \subseteq E^1 \subseteq K_i(E) \subseteq E$. Lastly we show that $C(E) \in F_i$ for all $i \in I$. Note that for all $n \geq 1$ and for all $\forall i \in N$, $C(E) \subseteq E^{n+1} \subseteq K_i(E^n)$. By propositions 3 and 5, we can show that $C(E) \subseteq \cap_{n \geq 1} K_i(E^n) \subseteq K_i(\cap_{n \geq 1} E^n) = K_i(C(E))$.

We are in a position to demonstrate the main result of this chapter "Aumann's impossibility of agreeing to disagree". We introduce a measurable function with endowed information partition to follow Aumann's main theorem.

Definition 4 : A function $f:\Omega \to R$ is measurable with respect to partition Π_i, if for all w, w' $\in \Omega$, $\Pi_i(w) = \Pi_i(w')$ implies $f(w) = f(w')$.

Proposition 7 : Let x_i for $i \in I$ be a collection of numbers, one for each agent, and let $C_i = \{w \in \Omega : E(f \mid \Pi_i)(w) = x_i\}$ be the event at which the expected value of f conditional on Π_i is x_i. Let $C = \cap_{i \in N} C_i$. If C is common knowledge at w, then $x_i = x_j$ for all $i, j \in I$.

Proof : Assume that C is common knowledge at w among the members of I. Then there exists an event $A \in F_i$ for all i such that $w \in A \subseteq C$. Note that $E(f \mid \Pi_i)(w) = \dfrac{\Sigma_{w' \in \Pi_i(w)} f(w')\mu(w')}{\mu(\Pi_i(w))}$, which can be written as $\Sigma_{w' \in \Pi_i(w)} E(f \mid \Pi_i)(w')\mu(w') = \Sigma_{w' \in \Pi_i(w)} f(w')\mu(w')$. This equation can hold when we substitute A for $\Pi_i(w)$ since A is some union of elements of information partition. Since we have $A \subseteq C$ and $E(f \mid \Pi_i)(w) = x_i$ for all $w \in C$. We obtain the following result: $\Sigma_{w \in A} E(f \mid \Pi_i)(w)\mu(w) = x_i\mu(A) = \Sigma_{w \in A} f(w)\mu(w)$, which implies that $x_i = x_j$ for all $i, j \in I$.

Suppose that the agents in the economy exchange their conditional expectation of some random variable each other given the true state of the economy. If this communication among agents in the economy is perfect in the sense of common knowledge, then the above proposition says that they can not agree to disagree.

IV. Common knowledge of an aggregate of expectation

4.1 The model

Let $X : \Omega \rightarrow R$ be a random variable. For each $i \in I$, let $X_i : \Omega \rightarrow R$ be conditional expectation of X on information partition Π_i. The value of conditional expectation of X at w is as follows.

$X_i = \dfrac{\Sigma_{w' \in \Pi_i(w)} X(w')\mu(w')}{\mu(\Pi_i(w))}$. This equation can be extended as $\sum_{w \in A} X_i(w)\mu(w) = \sum_{w \in A} X(w)\mu(w)$ for $all A \in F_i$ since A is some union of elements of information partition.

Let $f_i : R \rightarrow R$ for each $i \in I$ be a strictly increasing function, and let $f : R^n \rightarrow R$ be defined by $f(x_1, x_2, ..., x_n) = \Sigma_{i \in I} f_i(x_i)$. Here f is an aggregate of conditional expectations of each agent. Fix $y \in R$, define the event $E = \{w \in \Omega \mid f(X_1(w), ..., X_n(w)) = y\}$. The following proposition says that if E is commonly known event among agents, then they can not agree to disagree: that is, their conditional expectations of some economic variable should be the same.

Proposition 1 : Suppose E is common knowledge at w^*. Then the n conditional expectations of X at w^* are equal.

Proof: Since E is common knowledge at w^*, there exists an event $A \in F_i$ for all $i \in I$ such that. w^* belongs to A which is a subset of E. Let's define the conditional expectation of X given the common event A as follows. $X_0 = \dfrac{1}{\mu(A)} \Sigma_{w \in A} X(w)\mu(w)$. This equation can be written as

$$\Sigma_{w \in A} X_0 \mu(w) = \Sigma_{w \in A} X(w)\mu(w) = \Sigma_{w \in A} X_i(w)\mu(w). \quad (1)$$

Since by assumption $f(x_1,x_2,\ldots x_n)$ is constant on A, we have

$$\Sigma_{w\in A} f(X_1(w),\ldots,X_n(w))(X(w)-X_0)=0$$

By definition of f,

$$\Sigma_{w\in A}\Sigma_{i\in I} f_i(X_i(w))(X(w)-X_0)\mu(w)=\Sigma_{i\in I}\Sigma_{w\in A} f_i(X_i(w))(X(w)-X_0)\mu(w)=0$$

Following the same procedure as above , We can also get

$$\Sigma_{i\in I}\Sigma_{w\in A} f_i(X_i(w))(X_i(w)-X_0)\mu(w)=0 \qquad (2)$$

Using (1), we get the following equation for all $i\in I$:

$\Sigma_{w\in A} f_i(X_0)(X_i(w)-X_0)\mu(w)=0$ and hence

$$\Sigma_{i\in N}\Sigma_{w\in A} f_i(X_0)(X_i(w)-X_0)\mu(w)=0 \qquad (3)$$

Combining equation (2) and (3) we get

$$\Sigma_{i\in I}\Sigma_{w\in A}(f_i(X_i(w)-f_i(X_0))(X_i(w)-X_0)\mu(w)=0.$$

Since f_i is strictly monotone, the above equality can be satisfied only if $X_i(w)=X_0$ for all $i\in I$ and all $w\in A$.

The assumption that f is monotone without being additive separable is not sufficient for the above proposition to hold. To see this, consider the following example

Let $\Omega=\{w_1,w_2,\ldots w_5\}$ be the set of states of the world, each with probability 1/5, and let I={1,2,3,4,5,6} be the set of agents. The agents information partitions are given by

$$\Pi_1=\{\{w_1,w_3\},\{w_2,w_4,w_5\}\}$$
$$\Pi_2=\{\{w_1,w_4\},\{w_2,w_3,w_5\}\}$$
$$\Pi_3=\{\{w_1,w_5\},\{w_2,w_3,w_4\}\}$$
$$\Pi_4=\{\{w_2,w_3\},\{w_1,w_4,w_5\}\}$$
$$\Pi_5=\{\{w_2,w_4\},\{w_1,w_3,w_5\}\}$$
$$\Pi_6=\{\{w_2,w_5\},\{w_1,w_3,w_4\}\}.$$

Consider the random variable $X:\Omega\to R$ given by

$X(w)=1$ if $w=w_1,w_2$, otherwise $X(w)=0$.

The conditional expectations, X_i of the agents are given by

	w_1	w_2	w_3	w_4	w_5
X_1	1/2	1/3	1/2	1/3	1/3
X_2	1/2	1/3	1/3	1/2	1/3
X_3	1/2	1/3	1/3	1/3	1/2
X_4	1/3	1/2	1/2	1/3	1/3
X_5	1/3	1/2	1/3	1/2	1/3
X_6	1/3	1/2	1/3	1/3	1/2

Consider the function $f: R^6 \rightarrow R$ given by $f(x_1,\ldots,x_6)=0$. This function is monotone in the domain of the possible 5 vectors of expectations given by the above table. It is common knowledge that $f(x_1,\ldots,x_6)=0$. Still, the conditional expectations are not equal.

References

Aumann, R. 1976. Agreeing to Disagree. Annals of Statistics 4: 1236-1239.

Grimmett G.R. and D.R. Stirzaker 1992. Probability and Random Processes, second edition, Oxford University.

Lars Tyge Nielson, Adam Brandenburger, John Geanakoplos, and Richard McKelevery, and Talbot Page, 1990. Common knowledge of an Aggregate of Expectation, 135-1239

Monderer, D, abd D.Samet. 1989. Approximating common knowledge with common beliefs. Games and Economic Behavior 1: 170-190.