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Linear Operators That Preserve Maximal Column Rank of Fuzzy Matrices

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Abstract

For each $m \geq 2$ and $n \geq 3$, we characterize the linear operators, T, on the set of $m \times n$ fuzzy matrices that preserve maximal column rank. That is, T preserves maximal column ranks if and only if T strongly preserves maximal column rank 1 and it preserves maximal column rank 3. Other characterizations of maximal column rank-preserving operators are also given.

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1 Introduction

There are many papers on the study of linear operators that preserve the semiring rank and the column rank of matrices over several semirings. Beasely and Pullman [2] obtained characterization of linear operators that preserve semiring rank of fuzzy matrices. Song [6] characterized the column rank case.

In this paper, we study the extent to which known properties of linear operators preserving the semiring ranks and the column ranks of matrices over 'chain semiring' (see Section 2) carry over to operators preserving maximal column ranks. We obtain some characterizations of linear operators that preserve maximal column rank of fuzzy matrices and of matrices over chain semirings which is more general than the Boolean algebra.

2 Preliminaries

A *semiring* is a binary system $(m \times n, +, \times)$ such that $(\mathbb{S}, +)$ is an Abelian monoid (identity 0),$(\mathbb{S}, \times)$ is a monoid (identity 1), $\times$ distributes over $+$, $0 \times s = s \times 0 = 0$ for all s in $\mathbb{S}$ and $1 \neq 0$. Usually $\mathbb{S}$ denotes both the semiring and the set and $\times$ is denoted by juxtaposition.

Let the set of $m \times n$ matrices with entries in a semiring $\mathbb{S}$ be denoted by $\mathbb{M}_{m,n}(\mathbb{S})$. The zero matrices $0_{m,n}$ and the $n \times n$ identity identity matrix I_n are defined as if $\mathbb{S}$ were a field. Addition, multiplication by scalars, and the product of matrices are also defined as if $\mathbb{S}$ were a field. If $\mathbb{V}$ is a nonempty subset of $\mathbb{S}^k \equiv \mathbb{M}_{k,1}(\mathbb{S})$ that is closed under addition and multiplication by scalars, then $\mathbb{V}$ is called a *vector space* over $\mathbb{S}$. The notions of subspace and of spanning or generating sets are the same as if $\mathbb{S}$ were a field. We will use the notation $< \mathbb{W} >$ to denote the subspace spanned by the subset of $\mathbb{W}$ of $\mathbb{V}$. A set G of vectors over $\mathbb{S}$ is *linearly dependent* if for some $g \in G$, $g \in < G \backslash \{g\} >$. Otherwise, G is *linearly independent.* The *maximal column rank*, $mc(A) = mc_{\mathbb{S}}(A)$, of an $m \times n$ matrix A over $\mathbb{S}$ is the maximal number of the columns of A which are linearly independent over $\mathbb{S}$. As with fields, a basis for a vector space $\mathbb{V}$ is a generating subset of the least cardinality. That cardinality is the *dimension* , $dim(\mathbb{V})$, of $\mathbb{V}$. The *column space* of an $m \times n$ matrix A over $\mathbb{S}$ is the vector space spanned by its columns. The *column rank*, $c(A) = c_{\mathbb{S}}(A)$, of an $m \times n$ matrix A over $\mathbb{S}$ is the dimension of the column space. The *semiring rank* of a nonzero matrix A in $\mathbb{M}_{m,n}(\mathbb{S})$ is the least integer k such that $A = BC$ for some $m \times k$ and $k \times n$ matrices B and C over $\mathbb{S}$. The *semiring rank* of the zero matrix is 0. we denote the semiring rank of A by $r(A)$ or $r_{\mathbb{S}}(A)$.

It follows directly from the definitions that for all $m \times n$ matrices A over $\mathbb{S}$:

(2.1) $0 \leq r_{\mathbb{S}}(A) \leq c_{\mathbb{S}}(A) \leq mc_{\mathbb{S}}(A) \leq n$;

(2.2) The semiring rank of a nonzero matrix A is the minimum number of semiring rank 1 matrices which sum to A([2, Lemma 2.1]).

Let $\mathbb{S}$ be any set of two or more elements. If $\mathbb{S}$ is totally odered by $<$, that is, $\mathbb{S}$ is a chain under $<$ (i.e. $x < y$ or $y < x$ for all distinct x, y in $\mathbb{S}$), then define $x + y = max(x, y)$ and $xy = min(x, y)$ for all x, y in $\mathbb{S}$. If $\mathbb{S}$ has a universal lower bound and a universal upper bound, then $\mathbb{S}$ becomes a semiring; a *chain semiring.*

Let H be any nonempty family of sets ordered by inclusion, $0 = \bigcap_{x \in H} x$, and $1 = \bigcup_{x \in H} x$. Then $\mathbb{S} = H \cup \{0, 1\}$ is a chain semiring. Let α, ω be real numbers with $\alpha < \omega$. Define $\mathbb{S} = \{\beta : \beta \in [\alpha, \omega]\}$. Then $\mathbb{S}$ is a chain semiring with $\alpha ='' 0''$ and $\omega ='' 1''$. It is isomorphic to the chain semiring in the previous example with $H = \{[\alpha, \beta] : \alpha \leq \beta \leq \omega\}$. If in particular we choose the real numbers 0 and 1 as α and ω in the previous example, then each $m \times n$ matrix over $\{\beta : 0 \leq \beta \leq 1, \ \ \beta$ is real$\}$ is the *fuzzy matrix.*

It is already known that:

(2.3) The column rank of a matrix over a chain semiring is unchanged by pre-or post-multiplication by an invertible matrix. Furthermore, the column rank of a 2×2 matrix is unchanged by transposition([6, Lemma 2.1]).

If we take to be a singleton, say $\{a\}$, and denote the empty set by 0 and $\{a\}$ by 1, the resulting chain semiring is merely the binary Boolean algebra, and denoted by $\mathbb{B}$.

Hereafter, otherwise specified, $\mathbb{K}$ will denote a chain semiring which is not the binary Boolean algebra , all matrices will denote the $m \times n$ matrices over a chain semiring and we will write $\mathbb{M}$ or $\mathbb{M}(\mathbb{K})$ for $\mathbb{M}_{m,n}(\mathbb{K})$.

Beasely and Pullman [3] obtain the following relation between semiring rank and column rank over $\mathbb{M}_{m,n}(\mathbb{K})$ and $\mathbb{M}_{m,n}(\mathbb{B})$.

Theorem 2.1. *([3, Theorem 2 and 3]) Let $\mu(\mathbb{S}, m, n)$ be the largest integer k such that for all $m \times n$ matrices A over $\mathbb{S}$, $r(A) = c(A)$ if $r(A) \leq k$ and $\alpha(\mathbb{S}, m, n)$ be the largest integer k such that for all $m \times n$ matrices A over $\mathbb{S}$, $c(A) = mc(A)$ if $c(A) \leq k$. Then*

(1) for any chain semiring $\mathbb{K}$, we have

$$\mu(\mathbb{K}, m, n) = \begin{cases} 2 & \text{if } m \geq 2 \text{ and } n = 2, \\ 1 & \text{otherwise.} \end{cases}$$

(2) for the binary Boolean algebra $\mathbb{B}$,

$$\mu(\mathbb{B}, m, n) = \begin{cases} 1 & \text{whenever } \min(m, n) = 1, \\ 3 & \text{for all } m \geq 3 \text{and } n = 3, \\ 2 & \text{otherwise.} \end{cases}$$

We give the following example for Theorem 2.2.

Example 2.1. *Let p be a nonzero nonunit element of $\mathbb{K}$. Consider,*

$$A = \begin{pmatrix} p & 0 & 1 \\ 0 & p & 1 \end{pmatrix}.$$

Since all the three columns of A are linearly independent, $mc(A) = 3$. But $c(A) = 2$, because $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ generates $< A >$.

Lemma 2.1. *([4, Lemma 2.2]) Over any semiring* $\mathbb{S}$, *if* $mc(A) > c(A)$ *for some* $p \times q$ *matrix* A, *then for all* $m \geq p$ *and* $n \geq q$, $\alpha(\mathbb{S}, m, n) < c(A)$.

Theorem 2.2. *Let* $\mathbb{K}$ *be a chain semiring. Then we have*

$$\alpha(\mathbb{K}, m, n) = \begin{cases} 2 & \text{if } m = n = 2, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Consider the matrix A in Example 2.1. Then by lemma 2.1, we may conclude that

$$\alpha(\mathbb{K}, m, n) \leq 1 \quad \text{if m} \geq 2 \text{ and n} \geq 3.$$

If $c(B) = 1$, then clearly $mc(B) = 1$, for any matrix over $\mathbb{K}$. Thus ,

$$\alpha(\mathbb{K}, m, n) = 1 \quad \text{if m } \geq 2 \text{ and n} \geq 3.$$

Suppose $m = n = 2$ and $c(A) = 2$. If $mc(A) = 1$, one column, say first, is the scalar multiple of the second column. But this is impossible, because $c(A) = 2$. Hence ,

$$\alpha(\mathbb{K}, m, n) = 2 \quad \text{if m} = \text{n} = 2.$$

It is trivial that $\alpha(\mathbb{K}, m, n) = 1$, for other values of m and n. □

Lemma 2.2. *([3]) If the columns of* $A \in \mathbb{M}_{m,n}(\mathbb{B})$ *are linearly independent, then* $mc(A) = c(A) = n$.

Since 1 is the only invertible member of the multiplicative monoid of $\mathbb{K}$, the permutation matrices (obtained by permuting the columns of I_n) are the only invertible members of $\mathbb{M}_{n,n}(\mathbb{K})$.

Lemma 2.3. *The maximal column rank of a matrix is unchanged by pre- or post-multiplication by an invertible matrix. Furthermore, the maximal column rank of a* 2×2 *matrix is unchanged by transposition.*

Proof. The results follow from Theorem 2.2 using (2.3). □

A function T mapping $\mathbb{M}_{m,n}(\mathbb{S})$ into $\mathbb{M}_{m,n}(\mathbb{S})$ called an *operator* on $\mathbb{M}_{m,n}(\mathbb{S})$. The operator T

(i) is *linear* if $T(\alpha A + \beta B) = \alpha T(A) + \beta T(B)$ for all $\alpha, \beta \in \mathbb{S}$ and all $A, B \in M_{m,n}(\mathbb{S})$,

(ii) *preserves semiring rank h* if, for any $A \in \mathbb{M}_{m,n}(\mathbb{S})$ with $r(A) = h$, $r(T(A)) = r(A)$,

(iii) *preserves column rank k* if, for any $A \in \mathbb{M}_{m,n}(\mathbb{S})$ with $c(A) = k$, $c(T(A)) = c(A)$,

(iv) *preserves maximal column rank l* if, for any $A \in \mathbb{M}_{m,n}(\mathbb{S})$ with $c(A) = l$, $mc(T(A)) = mc(A)$,

(v) is a *congruence operator* if there exist invertible martices U and V in $\mathbb{M}_{m,m}(\mathbb{S})$ and $\mathbb{M}_{n,n}(\mathbb{S})$ respectively such that $T(A) = UAV$ for all $A \in \mathbb{M}_{m,n}(\mathbb{S})$

(vi) is a *transposition operator* if $m = n$ and $T(A) = A^t$ for all $A \in \mathbb{M}_{m,n}(\mathbb{S})$.

Lemma 2.4. *Congruence operators on $\mathbb{M}_{m,n}(\mathbb{F})$ are linear, bijective, and preserves all maximal column rank.*

Proof. Linearity follows from the linearity of matrix multiplication. The rest follows from Lemma 2.3. □

Let $\mathbf{j}_k$ denote the column vector of length k all of whose entries are 1, and J_{mn} the $m \times n$ martix all of whose entries are 1. When the orders are understood, we may drop the subscript on $\mathbf{j}_k$ and J_{mn}. Let E_{ij} be the $m \times n$ matrix all of whose entries are 0 except the (i, j)th, which is 1.

Let $X \in \mathbb{M}_{m,n}(\mathbb{K})$. The *norm* $\|X\|$ of X is defined by $\|X\| = \mathbf{j}^t X \mathbf{j}$ the sum of all entries in X. That is, $\|X\|$ is the maximum entry in X. Note the mapping $X \to \|X\|$ preserves matrix addition and scalar multiplication.

Lemma 2.5. *([6, Lemma 2.3]) Suppose*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then $c(A) = 2$ if and only if $ad \neq bc$.

Lemma 2.6. *Suppose*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then $mc(A) = 2$ if and only if $ad \neq bc$.

Proof. $mc(A) = 2$ if and only if $c(A) = 2$, by Theorem 2.2. The result follows from Lemma 2.5. □

Lemma 2.7. *If H is a submatrix of A, then $mc(H) \leq mc(A)$.*

Proof. It is clear from definition of the maximal column rank. □

3 Linear Operators that preserve maximal column rank over $\mathbb{M}_{m,n}(\mathbb{K})$

In this section, we characterize linear operators that preserve maximal column rank over $\mathbb{M}_{m,n}(\mathbb{K})$. Hereafter, we shall adopt the convention $m \leq n$, and the set of matrices of maximal column rank 1 over a fixed chain semiring $\mathbb{K}$ is denoted by C_1. Two maximal column rank 1 matrices A, B are said to be *separable* if there is a matrix X with $mc(X) = 1$ such that either $1 = mc(A + X) < mc(B + X)$ or $1 = mc(B + X) < mc(A + X)$. In this case, X is said to separate A from B.

Using Theorem 2.2 we can apply some results in [6] for column rank 1 matrices to those for maximal column rank 1 matrices. Thus we obtain the following Theorem 3.1 by the analogue proof of that in [6].

Theorem 3.1. *Distinct maximal column rank 1 matrices are separable if and only if at least one of them is not a scalar multiple of J.*

The symbol $\leq$ is read entrywise, i.e. $X \leq Y$ if and only if $x_{ij} \leq y_{ij}$ for all (i, j).

Lemma 3.1. *([2, Lemma 4.3])*
If T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{K})$, $min(m, n) > 1$, T preserves norm, and $A \leq T(A)$, then $T^q(A) = T^{mn-1}(A)$ for all $q \geq mn$.

Lemma 3.2. *Let T be a linear operator on $\mathbb{M}_{m,n}(\mathbb{K})$ with $min(m, n) > 1$. If T preserves norm and maximal column rank 1 but is not injective on C_1, then T reduces the maximal column rank of some matrix from $k(\geq 2)$ to 1.*

Proof. Since T is not injective on C_1, $T(A) = T(B)$ for some A, B in C_1 with $A \neq B$. If $A = \alpha J$ and $B = \beta J$, then $\alpha = \beta$ because T preserves norms, contradicting our assumption that $A \neq B$. Therefore by Theorem 3.1, some matrix X of maximal column rank 1 separates A from B. Say, $mc(A+X) = 1$ and $mc(X + B) = k \geq 2$. Since

$$T(X + B) = T(X) + T(B) = T(X) + T(A) = T(X + A),$$

T reduces the maximal column rank of $X + B$ from k to 1. □

We say that a linear operator T on $\mathbb{M}_{m,n}(\mathbb{K})$ *strongly preserves maximal column rank 1*, provided that $mc(X) = 1$ if and only if $mc(T(X)) = 1$.

Lemma 3.3. *If T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{K})$ $min(m,n) > 1$, and T strongly preserves maximal column rank 1, then T preserves norm.*

Proof. Let $A \in \mathbb{M}_{m,n}(\mathbb{K}), \alpha = \|A\|$ and $\beta = \|T(A)\|$; then $A = \alpha A$ and $\beta = \|T(A)\| = \|T(\alpha A)\| = \alpha\|T(A)\| \leq \alpha$. Suppose $\beta < \alpha$. Then for some (i,j), $a_{ij} = \alpha$. Let Y be the matrix whose entries are all α except for $y_{ij} = 0$. Then $\alpha J = A + Y$. So $mc(A+Y) = 1$. Since $mc(\beta A + Y) \geq 2$ by Lemma 2.6 and Lemma 2.7, but $mc(\beta A + Y) \leq 2$ by construction, we have $mc(\beta A + Y) = 2$. By linearity of T and definition of β, we have $T(\beta A) = \beta T(A) = T(A)$. Hence $T(\beta A + Y) = T(\beta A) + T(Y) = T(A) + T(Y) = T(A+Y) = \alpha T(J)$. So T reduces the maximal column rank of $\beta A + Y$ from 2 to 1, contrary to our hypothesis. Thus T preserves norm. □

Lemma 3.4. *Suppose T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{K})$ and $min(m,n) >$ 1. If T strongly preserves maximal column rank 1, then T permutes Γ, where $\Gamma = \{E_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$*

Proof. By Lemma 3.3, T preserves norm. Therefore by Lemma 3.2, T is injective on C_1. Suppose $T(E_{pq})$ is not in Γ for some (p,q). Now $T(E_{pq}) = \sum \tau_{ij} E_{ij}$, for some τ_{ij}. But $\|T(E_{pq})\| = 1$, so $\tau_{uv} = 1$ for some (u,v). Without loss of generality, we may assume that $(u,v) = (p,q)$, because if P, Q are permutation matrices, then the linear operator $X \to PT(X)Q$ preserves the maximal column ranks that T preserves (see Lemma 2.3.) and permutes Γ if and only if T does. Let $E = E_{pq}$. Then $E \leq T(E)$, so $E \neq T(E) \leq T^2(E) \leq \cdots \leq T^k(E) = T^{k+h}(E)$, where k is the least integer for which equality holds and $h \geq 0$ is arbitrary. By Lemma 3.1, we are assured that k exists and is less than mn. Let $B = T^{k-1}(E)$. Then $B \neq T(B)$ but $T(B) = T(T(B))$, despite the fact that B, $T(B)$ are both in C_1 and T is injective on C_1. This contradiction implies that T maps Γ into Γ. By injectivity, T permutes Γ. □

Let $\mathbb{B}$ be the two element subsemiring $\{0,1\}$ of $\mathbb{K}$, and α be a fixed member of $\mathbb{K}$, other than 1. For each x in $\mathbb{K}$ define $x^\alpha = 0$ if $x \leq \alpha$, and $x^\alpha = 1$ otherwise. Then the mapping $x \to x^\alpha$ is a homomorphism of $\mathbb{K}$ onto $\mathbb{B}$. Its entrywise extension to a mapping $A \to A^\alpha$ of $\mathbb{M}(\mathbb{K})$ onto $\mathbb{M}(\mathbb{B})$ preserves matrix sums and products and multiplication by scalars. We call A^α the $\alpha - pattern$ of A.

Example 3.1. *For a nonzero nonunit* $p \in \mathbb{K}$, *consider*

$$A = \begin{pmatrix} p & p & p \\ p & p & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then $mc(A) = 3$, *because all the three columns of* A *are linearly independent. But* $mc(A^t) = 2$. *Consider* $B = A \oplus 0_{m-3,m-3}$ *for all* $m \geq 3$. *If* T *is a transposition operator over* $\mathbb{M}_{m,m}(\mathbb{K})$, *then* $T(B) = B^t$ *has maximal column rank 2 while* $mc(B) = 3$. *Thus a transposition operator does not preserve maximal column rank 3.*

Earlier, linear operators that preserve semiring rank and column rank over $\mathbb{M}(\mathbb{K})$ were characterized in [2] and [6], respectively. Also linear operators that preserve maximal column rank preserving operators over $\mathbb{M}_{m,n}(\mathbb{B})$ were characterized in [4]. For our purpose, we write those results as follows;

Lemma 3.5. *(1) ([2, Theorem 4.2]) Suppose* T *is a linear operator on* $\mathbb{M}_{m,n}(\mathbb{K})$ *with* $n \geq m \geq 1$. *Then* T *is bijective and preserves semiring rank 1 if and only if it is in the group of operators generated by congruence and transposition operators.*

(2) ([6, Theorem 3.3]) Suppose T *is a linear operator on* $\mathbb{M}_{m,n}(\mathbb{K})$ *with* $m \geq 2$ *and* $n \geq 3$. *Then* T *strongly preserves column rank 1 and it preserves column rank 3 if and only if it is a congruence operator.*

(3) ([4, Theorem 3.2]) Suppose T *is a linear operator on* $\mathbb{M}_{m,n}(\mathbb{B})$ *for* $n \geq m \geq 4$. *Then* T *preserves maximal column ranks 1, 2 and 3 if and only if it is a congruence operator. Moreover the transposition operator on* $\mathbb{M}_{m,m}(\mathbb{B})$ *does not preserve maximal column rank 3 for* $m \geq 4$.

We say that an $m \times n$ matrix X is a *column matrix* if $X = \mathbf{x}(\mathbf{e}_i)^t$ for some $\mathbf{x} \in \mathbb{S}^m$ and $\mathbf{e}_i \in \mathbb{S}^n$, where $\mathbf{e}_i$ is the vector with 1 in the ith position and 0 elsewhere.

Theorem 3.2. *Suppose* T *is a linear operator on the* $m \times n$ *matrices over a chain semiring* $\mathbb{K}$, *where* $m \geq 2$ *and* $n \geq 3$. *If* T *strongly preserves maximal column rank 1, and it preserves maximal column rank 3, then* T *is a congruence operator.*

Proof. Let $\bar{\mathbb{M}} = \mathbb{M}_{m,n}(\mathbb{B})$. Lemma 3.4 and linearity imply that T maps $\bar{\mathbb{M}}$ into itself. Let $\bar{T}$ denote the restriction of T to $\bar{\mathbb{M}}$. From the definition of maximal column rank, the maximal column rank $mc_{\mathbb{B}}(X)$ of a member X of $\bar{\mathbb{M}}$ is at least $mc_{\mathbb{K}}(X)$, its maximal column rank as a member of $\mathbb{M}_{m,n}(\mathbb{K})$, because $\mathbb{B} \subseteq \mathbb{K}$. On the other hand, the mapping that takes a matrix A in $\mathbb{M}_{m,n}(\mathbb{K})$ to its *0-pattern* A^0 in $\bar{\mathbb{M}}$ preserves matrix sums and multiplication by scalars. Hence $mc_{\mathbb{B}}(X) = mc_{\mathbb{K}}(X)$ for all X in $\bar{\mathbb{M}}$. Therefore $\bar{T}$ strongly preserves maximal column rank 1, and it preserves maximal column rank 3.

Case 1 ($n \geq m \geq 4$). Since $\bar{T}$ also permutes Γ by Lemma 3.4 and it strongly preserves maximal column rank 1, $\bar{T}$ must map a column matrix either a column matrix or transpose of a column matrix if $m = n \geq 4$. For the latter case, $\bar{T}$ is a composition of a transposition operator and pre-multiplication by a permutation matrix. Since transposition operator cannot preserve maximal column rank 3 by Lemma 3.5(3), $\bar{T}$ must map a column matrix to a column matrix. Thus the linearity of $\bar{T}$ implies that $mc_{\mathbb{B}}(\bar{T}(X)) \leq mc_{\mathbb{B}}(X)$ for all X in $\bar{\mathbb{M}}$. In particular, $\bar{T}$ preserves maximal column rank 2. Hence $\bar{T}$ is a congruence operator on $\bar{\mathbb{M}}$ by Lemma 3.5(3). Then $\bar{T}(X) = UXV$ for some invertible matrix over $\mathbb{M}$. Notice that matrices U, V are also invertible over $\mathbb{K}$; in fact, they are just permutation matrices. Let $A \in \mathbb{M}(\mathbb{K})$. Then $T(A) = \sum a_{ij}T(E_{ij}) = \sum a_{ij}\bar{T}(E_{ij})$, because each E_{ij} is in $\bar{\mathbb{M}}(\mathbb{B})$. Since $\bar{T}(E_{ij}) = UE_{ij}V$ for all i, j by definition of congruence operator, the result follows directly from the linearity of matrix multiplication.

Case 2($n = 3$ and $2 \leq m \leq 3$). Theorem 2.2 guarantees that $\bar{T}$ strongly preserves column rank 1. Note that $mc_{\mathbb{B}}(X) = 3$ if and only if $c_{\mathbb{B}}(X) = 3$ by Lemma 2.2 and (2.1). Hence it preserves column rank 3, because if $c_{\mathbb{B}}(X) = 3$, then $3 = mc_{\mathbb{B}}(X) = mc_{\mathbb{B}}(\bar{T}(X)) = c_{\mathbb{B}}(\bar{T}(X))$. Also, $\bar{T}$ strongly preserves semiring rank 1 and it preserves semiring rank 3, by Theorem 2.1(2). If $r_{\mathbb{B}}(X) = 2$ for $X \in \bar{M}$, then X can be factored as a sum of two matrices X_1 and X_2 whose semiring ranks are 1, by (2.2). Thus $\bar{T}(X) = \bar{T}(X_1) + \bar{T}(X_2)$ has semiring rank two or less. Since $\bar{T}$ strongly preserves semiring rank 1, $r_{\mathbb{B}}(\bar{T}(X)) = 2$. That is, $\bar{T}$ preserves semiring rank 2. Therefore $\bar{T}$ is in the group of operators generated by congruence (and if $m = n = 3$, also the transposition) operators by Lemma 3.5(1). Let $A \in M$. Then $T(A) = \sum a_{ij}T(E_{ij}) = \sum a_{ij}\bar{T}(E_{ij})$, since each E_{ij} is in $\bar{M}$. By similar argument as in case 1, there are permutation matrices U and V ($m \times m$ and $n \times n$ respectively) such that in the case $n = 3$ and $m = 2$, $T(A) = UAV$, while in the case $m = n = 3$, $T(A)$ is either UAV or UA^tV. However, since transposition operator does not preserve maximal column rank 3 by Example 3.1, we see that in fact, T must be a congruence operator. □

Theorem 3.3. *Suppose T is a linear operator on the $m \times n$ matrices over a chain semiring with $m \geq 2$ and $n \geq 3$. Then the following statements are equivalent:*

(i) T preserves all maximal column ranks.

(ii) T strongly preserves maximal column rank 1 and it preserves maximal column rank 3.

(iii) T is a congruence operator.

(iv) T is bijective and preserves maximal column ranks 1 and 3.

Proof. It is obvious that (i) implies (ii). Theorem 3.2 establishes that (ii) implies (iii). According to Lemma 2.4, (iii)implies (i) and (iv). If T satisfies (iv), then T is in the group of operators generated by congruence and transposition operators by Lemma 3.5(1) and Theorem 2.2. Since the transposition operator does not preserve maximal column rank 3, T must be a congruence operator. Therefore, (iv) implies (iii). □

How necessary is it that $m \geq 2$ and $n \geq 3$? If $m = n = 2$, then a linear operator that preserves all maximal column ranks is the same as a linear operator that preserves all column ranks by Theorem 2.2. The characterizations of the column rank preservers were obtained in [6]. Thus we have characterizations of the linear operators that preserve the maximal column rank of matrices over a chain semiring(and in particular, of fuzzy matrices) when $m \geq 2$ and $n \geq 3$.

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