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KRONECKER PRODUCTIVE PROPERTY OF POSITIVE DEFINITE MATRICES

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Abstract

For an $n \times n$ Hermitian matrix A, A is called ***positive definite*** if $x^*Ax > 0$ for all nonzero vector x of order n over complex field. In this paper, we consider the Kronecker product of positive definite matrices. We prove that the Kronecker product of positive definite matrices is also positive definite.

Keywords: Positive definite matrix, Kronecker product, Schur product.

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1 Introduction

A class of Hermitian matrices with a special positivity property arises naturally in many applications. Hermitian and real symmetric matrices with this positive property also provide one generalization to matrices of the notion of positive number. This observation often provides insight into the properties and applications of positive definite matrices. These matrices arise in many applications : in harmonic analysis, in complex analysis, in the theory of vibrations of mechanical systems, and in other areas of matrix theory such as the singular value decomposition and the solutions of linear least-squares problems. So there are many papers on the matrix theory ([1]-[5]), in particular, on the positive definite matrices. In this paper, we consider the productive properties of two positive definite matrices. That is the Kronecker product of positive definite matrices also positive definite? For this question, we give a positive answer by a new proof. We also have some related properties of positive definite matrices.

2 Preliminaries and definitions

We give some definitions and known results on the positive definite matrix and Kronecker product.

Let $\mathbb{M}$ denote the set of $n \times n$ matrices with entries from the complex field $\mathbb{C}$.

DEFINITION 2.1. A matrix $A = [a_{ij}] \in \mathbb{M}$ is said to be *Hermitian* if $A = A^*$, where $A^* = \bar{A}^T = [\bar{a}_{ij}]$. It is *skew-Hermitian* if $A = -A^*$. It is *unitary* if $AA^* = I = A^*A$, where I is the identity matrix of order n.

Some observations for $A, B \in \mathbb{M}$:
1. $A + A^*$, AA^* and A^*A are all Hermitian for all $A \in \mathbb{M}$.
2. If A is Hermitian, then A^k is Hermitian for all $k = 1, 2, 3, \cdots$. If A is nonsingular as well, then A^{-1} is Hermitian.
3. If A, B are Hermitian, then $aA + bB$ is Hermitian for all real scalars a, b.
4. If A is Hermitian, the main diagonal entries of A are all real numbers.

THEOREM 2.2.([2] Schur's Theorem) Given $A \in \mathbb{M}$ with eigenvalues $\lambda_1, \cdots, \lambda_n$ in any prescribed order, there is a unitary matrix $U \in \mathbf{M}$ such that

$$U^*AU = T = [t_{ij}]$$

is upper triangular, with diagonal entries $t_{ij} = \lambda_i, i = 1, \cdots, n$. That is, every square matrix A is unitarily equivalent to a triangular matrix whose diagonal entries are the eigenvalues of A in a prescribed order. □

DEFINITION 2.3. ([3]) A matrix $A \in \mathbb{M}$ is said to be *positive definite* if

$$x^* Ax > 0 \quad \text{for all nonzero} \quad x \in \mathbb{C}^n. \tag{2.1}$$

If the strict inequality required in (2.1) is weakened to $x^* Ax \geq 0$, then A is said to be *positive semidefinite*.

Implicit in these defining inequalities is the observation that if A is Hermitian, the left-hand side of (2.1) is always a real number. Of course, if A is positive definite, then it is also positive semidefinite. If A is positive definite, then A is Hermitian, but not conversely. A Hermitian matrix is positive definite if and only if all of its eigenvalues are positive.

DEFINITION 2.4. Let $A, B \in \mathbf{M}$ be Hermitian matrices. Then we write $A \succeq B$ if the matrix $A - B$ is positive semidefinite. Similarly, $A \succ B$ means that $A - B$ is positive definite.

Some observations for $A, B \in \mathbf{M}$:

5. $A \succeq B$ and $B \succeq A$ implies $A = B$.
6. Relation $\succeq$ is a partial order. That is, the relation $\succeq$ is transitive, reflexive but not a total order.
7. If A, B are Hermitian, then $A \succeq B$ implies $T^* AT \succeq T^* BT$ for all $T \in \mathbb{M}$.
8. If $A \succ B \succ 0$, then $det A > det B$ and $tr(A) > tr(B)$.

LEMMA 2.5.([4]) Suppose that $P \in \mathbb{M}$ is positive definite, and let $S \subset \{1, 2, \cdots, n\}$ be an index set. Then

$$P^{-1}(S) \succeq [P(S)]^{-1}$$

where the left-hand side of this inequality is principal submatrix of P^{-1} determined by deletion of the rows and columns indicated by S, while the right-hand side is the inverse of the corresponding submatrix of P. □

DEFINITION 2.6. Let $A, B \in \mathbb{M}$. The *Schur (or Hadamard) product* of $A = [a_{ij}]$ and $[b_{ij}]$ is just their entrywise product $A \circ B = [a_{ij} b_{ij}]$.

The *Kronecker product* of A and B is a $n^2 \times n^2$ matrix defined by

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & \cdots & a_{2n}B \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1}B & a_{n2}B & \cdots & \cdots & a_{nn}B \end{pmatrix}.$$

If $A, B \in \mathbb{M}$, and if $S = \{1, n+2, 2n+3, 3n+4, \cdots, n^2\}$, then

$$A \circ B = (A \otimes B)(S). \tag{2.2}$$

If A and B are invertible, then $A \otimes B$ is invertible and $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

3 Kronecker product of positive definite matrices.

In this section, we prove that the Kronecker product of positive definite matrices are also positive definite. We have some relative properties of positive definite matrices.

LEMMA 3.1. For $A, B, C, D \in \mathbb{M}$,

$$(A \otimes B)(C \otimes D) = (AC \otimes BD)$$

Proof. Consider the $(i,j)-$block of $n^2 \times n^2$ matrix $(A \otimes B)(C \otimes D)$. Then the $(i,j)-$block of $(A \otimes B)(C \otimes D)$ is

$$\sum_{k=1}^{n} (a_{ik}B)(c_{kj}D) = \sum (a_{ik}c_{kj}BD),$$

which is the $(i,j)-$block of $(AC \otimes BD)$. □

LEMMA 3.2. For $A, B \in \mathbb{M}$, we have

$$(A \otimes B)^* = A^* \otimes B^*.$$

Proof. Consider the entries of both sides. Then

$$(A \otimes B)^* = [a_{ij}B]^* = [\bar{a}_{ji}B^*] = A^* \otimes B^*.\square$$

LEMMA 3.3. For unitary matrices U and $V \in \mathbf{M}$, $U \otimes V$ is also unitary.

Proof. Using Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned}
&(U \otimes V)^*(U \otimes V) \\
=&(U^* \otimes V^*)(U \otimes V) \\
=&(U^*U) \otimes (V^*V) = I_n \otimes I_n = I_{n^2}.
\end{aligned}$$

Thus $U \otimes V$ is a unitary matrix. □

THEOREM 3.4 Let $\sigma(A) = \{\lambda_i | 1 \leq i \leq n\}, \sigma(B) = \{\mu_j | 1 \leq j \leq n\}$. Then

$$\sigma(A \otimes B) = \{\lambda_i \mu_j | 1 \leq i \leq n, 1 \leq j \leq n\}.$$

Proof. By the Schur's theorem, there are unitary matrices U and $V \in M_n$ such that $U^*AU = T_1$ and $V^*BV = T_2$ with upper triangular matrices T_1 and T_2. Since

$$(A \otimes B)(C \otimes D) = (AC \otimes BD)(by \;\; Lemma \; 3.1)$$

and

$$(A \otimes B)^* = A^* \otimes B^*(by \;\; Lemma \; 3.2),$$

it follows that

$$T_1 \otimes T_2 = (U^*AU) \otimes (V^*BV) = (U \otimes V)^*(A \otimes B)(U \otimes V).$$

Since $U \otimes V$ is unitary from Lemma 3.3, it follows that

$$\sigma(A \otimes B) = \sigma(T_1 \otimes T_2) = \{\lambda_i \mu_j | 1 \leq i \leq n, 1 \leq j \leq n\}.\square$$

THEOREM 3.5. Let A and B be positive definite matrices in $\mathbb{M}_n$. Then $A \otimes B$ is also positive definite. That is, the Kronecker product of positive definite matrices is also positive definite.

Proof. Since A and B are positive definite, the eigenvalues of them are all positive. Hence the products of the eigenvalues of A and B are all positive. Therefore all eigenvalues of $A \otimes B$ are positive by Theorem 3.4. Thus $A \otimes B$ is a positive definite matrix. □

COROLLARY 3.6. The Schur (or Hadamard) product of positive definite matrices is also positive definite.

Proof. By Theorem 3.5, the Kronecker product of positive definite matrices is positive definite. Since the principal submatrix of a positive definite matrix is also positive definite, the Schur product of positive definite matrices is a positive definite matrix from the equation $A \circ B = (A \otimes B)(S)$ in (2.2). □

Thus we have obtained the Kronecker productive property and Schur productive property of positive definite matrices through new proofs. We also state that the contents of Lemmas and Theorems hold as well for $m \times n$ matrices over a complex field.

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