On the left end point of the real numbers

Han Chul-soon

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Ⅰ Introduction

We call R the set of real numbers and Q the set of rational numbers.

Definition 1. A set $J \subset R$ is called an open upper segment if

a) $J \neq \phi$ and $J \neq R$

b) for every $x \in J$, there exist a $y \in J$ such that $y < x$

c) if $x \in J$ and $y < x$ then $y \in J$

Definition 2. A point x will be called the left end point of an open upper segment J if

a) for every $y \in J$, $x < y$

b) if z is such that for every $y \in J$, $z < y$, then $z \leq x$.

We define open upper segments of rationals and their left end points in the same way as for reals.

Every open upper segment of rationals has not always their left end point.

But any open upper segment of reals must have a left end point. We shall prove this such as the following.

Ⅱ Main Theorem

Theorem : Every open upper segment of reals has a left end point.

Proof : Let J be an cpen upper segment of reals.

Let $W=\cup\{\alpha : \alpha \varepsilon J\}$.

First, we shall show that W is an open upper segment of rationals. Since $J\neq\phi$ by definition Ⅰ, there exist some $\alpha\neq\phi$ and then $W\neq\phi$. Since $J\neq R$ by definition Ⅰ, there exist $\delta\notin J$ such that $\delta \varepsilon R$.

Therefore, there exist $x\notin\delta$ and $x \varepsilon Q$.

Thus $x\notin W$ and then $W\neq Q$.

Let $x \varepsilon W$, then there is an $\alpha \varepsilon J$ such that $x \varepsilon \alpha$.

Since J is an open upper segment, and then for every $y>x$, $y \varepsilon \alpha$. Thus we have $y \varepsilon W$.

Moreover, if $x \varepsilon \alpha$ and $x>y$ then $y \varepsilon \alpha$.

Thus we have $y \varepsilon W$.

Therefore, W is an open upper segment of Q.

Secondly, we must show that W is a left end point of J.

If for every $\alpha \varepsilon J$, $\alpha\subset W$, then $W< \alpha$ ($\because W,\alpha$: open upper segments). If $\delta\notin J$, then $\delta>\alpha$ for every $\alpha \varepsilon J$.

Since $W=\cup\{\alpha : \alpha \varepsilon J\}$, $W\subset\delta$.

Thus $W\geq\delta$.

Therefore, W is a left end point of J.

We can rewrite this theorem as following :

Corollary : If non-void set $S\subset R$ is bounded below, then S has greatest upper bound.

Proof : Let $J=\{x : x \varepsilon R,\ x>y$ for some $y \varepsilon S\}$.

Since $S\neq\phi$, there is an $x_0 \varepsilon S$ such that $x_0<x_0+1$

This $x_0+1 \varepsilon J$ and so $J\neq\phi$.

Since S is bounded below, then there is an $M \varepsilon R$ such that for every $x \varepsilon S$, $x\geq M$.

Hence $M\notin J$, and so $J\neq R$.

Let $x \varepsilon J$ and $y < x$. There is an $x_1 \varepsilon S$ such that $x > x_1$. Let $y = \frac{x+x_1}{2}$, then $x > y > x_1$ and $y \varepsilon J$.

Furthermore, let $x \varepsilon J$ and $x < y$.

There is an $x_1 \varepsilon S$ such that $x > x_1$.

Hence $y > x > x_1$, then $y \varepsilon J$.

Therefore, J is an open upper segment of R and J has a left end point α.

Accorclingly, we shall prove that α is a greatest lower bound of S. If there is $x_0 \varepsilon S$ such that $\alpha > x_0$, then $\alpha \varepsilon J$. Sime J is an open upper segment, we have a contradctiction. Thus α is a lower bound of S. Let $y_0 > \alpha$, then $y_0 \varepsilon J$.

There is an $x_0 \varepsilon S$ such that $y_0 > x_0$.

Thus y_0 is not a lower bound of S.

Therefore, α is a greatest lower bound of S.

Similarly, we can prove that if non-void set $S \subset R$ is bounded above then S has a least upper bound.

Ⅱ Concluding Remark

The existence of real number is given by Dedekind's cut. This paper is to consider the open upper segment itself to be a substitute for its own left end point. We can see that this is an entirely natural approach when we agree that open upper segments are to be in one-one correspondence with their left end point, that is, every open upper segment is to have a left end point and distinct open upper segment must have different left end points. Accordingly, the set R of reals is defined to be the set of open upper segments of rationals (adding the left end point to correspond to those open upper segments which do not have left end point among the rationals).

Any real number will be considered by the set such as any open upper segment.

References

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實數의 左端点에 關하여

한 철 순

실수의 개념은 Dedekind(1831—1916), Weierstrass(1815—1897), Cantor(1845—1918), Cauchy(1789—1857)등에 依하여 多角的인 관점에서 研究되어 왔다. 그 후의 수학자들도 여러가지 측면에서 계속 연구를 했으며, 본 논문도 전자의 방법과는 다른 (근본적인 개념은 동일하지만) 관점에서 실수의 개념을 파악하고자 하였다.

실수라는 것은 切斷에 依하여 存在하는 것으로 생각된다. 이런 切斷에 依한 實數의 存在를 대개는 下組(open lower segment)에 依하여 설명되지만 特히 본 논문에서는 上組(open upper segment)에 依하여 存在한다고 主張하였다.