Other Proof of Berg's Theorem

by Chul-soon Han

抄　錄

A가 Separable Hilbert space에 작용하는 Normal operater 일때 A=diagonal operator+compact operator 인 것은 I.D. Berg에 의해서 發見되었다.

本論文은 이 사실에 對한 P.R. Halmos의 증명 方法에 동기를 얻어서 다른하나의 證明方法을 提示하였다. 또한 이것을 하기위하여 B.R. Gellaum의 結果도 새로운 관점에서 證明하였다.

1 Introduction. Let A be the C*-algebra generated by a commuting, countable family of bounded normal operators $\{A_n\}$ and the identity operator I on a Hilbert space H.

B.R. Gelbaum [6] gave a simple proof of the following theorem due to von Neumann: There is a single Hermitian operator B on H such that the C*-algebra B generated by B and I contains the C*-algebra A [9, 13].

There are at least three different proofs other than the von Neumann's original proof, mainly because there exist different formulations of the spectral theorem. In this note, we shall use the "spectral measure" version of the spectral theorem. We are motivated by a Halmos paper [7]. With only a slight modification, his argument also can be applied for the proof of the von Neumann theorem. But there, he uses the "multiplication version" of the spectral theorem.

2 The von Neumann Therorem.

Lemma 1. Let A be a Banach algebra. Then A is separable if and only if it is generated dy a countable family of elements.

Corollary 2. Let A be a C*-algebra generated by a countable family $\{A_n\}$ of operators and the identity operator I on a Hilbert space H, then A is separable.

Lemma 3. If A is a Banach space, then the unit ball of the norm dual of A, with respect to the weak* topology, is a metric space if and only if A is a separable Banach space [3].

Lemma 4. Let Λ be a non-empty Hausdorff Space. Then Λ is a compact metrizable space with respect to the topology if and only if it is a continuous image of the Cantor set Γ [8].

Lemma 5. Let φ be a continuous function on the Cantor set Γ onto a non-empty compact metric space Λ. Then there exists a Borel function ψ on Λ into Γ such that $\varphi o \psi = I_\Lambda$, the identity function on Λ.

Proof. If $z \in \Lambda$, the compact set $\varphi^{-1}(z)$ in Γ

attains its minimum, say, $x \in \varphi^{-1}(z)$. Define $\psi(z)=x$. Then it is easy to see that $\psi(z) \leq \lim \inf \psi(z_n)$, whenever $z_n \to z$, by using the facts $\varphi(\psi(z_n))=z_n$ and the continuity of φ. Therefore ψ is lower semi-confinuous [2], i.e. $\psi^{-1}(t,\infty)$ is open for each real t. Since ψ is bounded, it is now clear that ψ is uniformly approximated by step functions in Borel set of Λ. Hence ψ is also a Borel measurable function on Λ into Γ.

Lemma 6. Let Γ, Λ be topological spaces and Σ_1, Σ their Borel σ-fields respectively. Let φ be a continuous function on Γ onto Λ. Suppose there is a Borel funtion on (Λ,Σ) into (Γ,Σ_1) such that $\varphi o \psi = I_\Lambda$.

Suppose that $\{E(\delta): \delta \in \Sigma\}$ is a bounded, vector valued, additive set function on Σ. If we define $E_1(\delta_1)=E(\varphi(\delta_1))$, $\delta_1 \in \Sigma_1$. then

(i) E_1 becomes a bounded, vector valued, additive set function on Σ_1, and

(ii) for each f $\in B(\Lambda,\Sigma)$ and

$$\int_\Gamma (fo\varphi)(t)\ dE_1(t)=\int_\Lambda f(\lambda)dE(\lambda).$$

Proof. Let $\{f_m\}$ be Σ-step functions such that $f_m \to f$ uniformly on Λ.

Each f_m has the form

$$\lambda_1\chi_{\delta_1}+\lambda_2\chi_{\delta_2}+\cdots+\lambda_n\chi_{\delta_n}, \text{ where}$$

δ_i are disjoint sets of Σ such that $\bigcup_{i=1}^{n}\delta_i=\Lambda$. Put $\varphi^{-1}(\delta_i)=\sigma_i$, $i=1,2,\ldots,n$.

Then

$$\{(\lambda_1\chi_{\sigma_1}+\lambda_2\chi_{\sigma_2}+\cdots+\lambda_n\chi_{\sigma_n})\}$$

converges to $fo\varphi$ on Γ uniformly.

Indeed, if

$$|(\lambda_1\chi_{\delta_1}+\lambda_2\chi_{\delta_2}+\cdots+\lambda_n\chi_{\delta_n}-f)(\lambda)| \leq \varepsilon$$

for all $\lambda \in \Lambda$, then

$$|(\lambda_1\chi_{\sigma_1}+\lambda_2\chi_{\sigma_2}+\cdots+\lambda_n\chi_{\sigma_n}-fo\varphi)(s)|$$

$=|\lambda_i-f(\varphi(s))|$, $\varphi(s) \in \delta_i$, if, say, $s \in \sigma_i$, $\leq \varepsilon$.

Hence $fo\varphi \in B(\Gamma, \Sigma_1)$.

Under the circumstances,

$\lambda_1E(\delta_1)+\lambda_2E(\delta_2)+\cdots+\lambda_nE(\delta_n) \to \int_\Lambda f(\lambda)dE(\lambda)$

and

$\lambda_1E(\sigma_1)+\lambda_2E(\sigma_2)+\cdots+\lambda_n) \to \int_\Lambda (fo\varphi)(s)dE_1(s)$.

But $E_1(\sigma_i)=E(\varphi(\sigma_i))=E(\delta_i)$, since $\sigma_i=\varphi^{-1}(\delta_i)$.

Hence

$$\int_\Lambda f(\lambda)dE(\lambda)=\int_\Gamma (fo\varphi)(s)dE_1(s).$$

In this proof, we used the fact that if $\sigma \in \Sigma_1$, then $\varphi(\sigma) \in \Sigma$.

Indeed, for $\sigma \in \Sigma_1$, $\psi^{-1}(\sigma) \in \Sigma$. But $\varphi o \psi = I_\Lambda$, so that

$$\varphi(\sigma)=(\varphi o \psi)\ (\psi^{-1}(\sigma))\ =\psi^{-1}(\sigma) \in \Sigma.$$

The relation that $\varphi(\sigma)=\psi^{-1}(\sigma)$ also gives the fact that $\sigma \to E_1(\sigma)=E(\varphi(\sigma))=E(\psi^{-1}(\sigma))$ is an addive set function. Q.E.D.

Theorem 7. Let $\{A_n\}$ be a countable family of commuting normal operators on a Hilbert space. Then there is a single Hermitian operator B and a sequence of a continuos functions $\{g_n\}$ on the spectrum Γ of B into the complex plane such that $A_n=\int_\Gamma g_n(t)dE_1(t)$, where $\{E_1(\delta_1): \delta_1 \in \Sigma_1\}$ denotes the spectral measure on the Borel σ-field Σ_1 in Γ, associated with B.

Proof. We apply corolly 2, Lemmas 3-6 and the general spectral theorem for a commutative C^*-algebra of operator [4]. Q.E.D.

Remark 1. The essential supremum norm as defined in [5] is equivalent to the following more intuitive definition.

$\|f\|_{ess}=\inf_{M>0} [M: E\{\lambda \in \Lambda: |f(\lambda)|>M\}=0]$, where

$f \varepsilon B(\Lambda, \Sigma)$ and E is the spectral measure for the commutative C*-algebra of operators with unit, whose spectrum is Λ.

Remark 2. Our proof is lengthier than that of Gelbaum, but looks more illuminating in relation with the general spectral theorem in the Dunford-Schwartz book metioned above.

3. An extension of Weyl-von Neumann Theorem.

Theorem 8. Let A be a Hermitian operator on a Hilbert space H with dim $H=\aleph_0$, then there is a compact operator C and an orthonormal basis for H such that

$$A = \begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & 0 \\ & & \lambda_3 & & & \\ & & & \lambda_4 & & \\ & & & & \ddots & \\ 0 & & & & & \end{pmatrix} + C$$

Furthermore C can be chosen to be a Hilbert-Schmit operator with arbitrary small Hilbert-Schmidt norm [i.e. the sum of the squares of the absolute values of the entries] [10].

Proof.

[I] If $e \varepsilon H$, $\|e\|=1$ and $\|(A-\lambda)e\| < \varepsilon \|e\|$, then

$$A = \left(\begin{array}{c|c} \lambda & 0 \\ \hline 0 & A_1 \end{array}\right) + C,$$

where C is a Hilbert-Schmidt operator with Hilbert Schmidt norm $<\sqrt{2}\,\varepsilon$.

Indeed, we write

$$A = \left(\begin{array}{c|c} a & X^* \\ \hline X & A_1 \end{array}\right) = \left(\begin{array}{c|c} \lambda & 0 \\ \hline 0 & A_1 \end{array}\right) + \left(\begin{array}{c|c} a-\lambda & X^* \\ \hline X & 0 \end{array}\right)$$

Now $\varepsilon^2 > \|(A-\lambda)e\|^2 = \|X\|^2_{HS}$, so

$$\|C\|^2_{HS} = \|X\|^2_{HS} + \|X^*\|^2_{HS} < 2\,\varepsilon^2.$$

[II] If $f \varepsilon H$, $\varepsilon > 0$, then there is a finite orthonormal set $\{e_1, e_2, e_3, \cdots, e_n\}$ such that $f \varepsilon \langle e_1, e_2, e_3 \cdots, e_n \rangle$ and

$$(1) \quad A = \begin{array}{c} e_1\ e_2\ e_3 \cdots\cdots e_n \\ \left(\begin{array}{c|c} \begin{matrix} \lambda_1 & & & & 0 \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ 0 & & & & \lambda_n \end{matrix} & 0 \\ \hline 0 & A_1 \end{array}\right) \end{array} + C$$

where C is a Hilbert-Schmidt operator with Hilbert-Schmidt norm $<\varepsilon$.

To prove this, let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ be positive real numbers such that

$$\Sigma_{i=1}^{n} \varepsilon_i > 2\|A\| \quad \text{and} \quad \Sigma_{i=1}^{n} \varepsilon_i^2 < \varepsilon^2/2$$

$$\begin{array}{c} |\leftarrow \varepsilon_1 \rightarrow|\ |\leftarrow \varepsilon_2 \rightarrow| \qquad |\leftarrow \varepsilon_n \rightarrow| \\ \text{——} [- Y_1 -)(- Y_2 -)(\text{——})(- Y_n -) \text{——} \\ -\|A\| \qquad\qquad\qquad\qquad \|A\| \end{array}$$

Let

$$Y_i = [-\|A\| + \sum_{j=1}^{i-1} \varepsilon_j,\ -\|A\| + \sum_{j=1}^{i} \varepsilon_j).$$

Then the spectrum

of $A, \sigma(A) \subset \bigcup_{i=1}^{n} Y_i$.

Hence $H = \Sigma_{i=1}^{n} \oplus H_i$, $H_i = E(Y_i \cap \sigma(A))H$, where E is the spectral measure associated with A. By neglecting Y_i's for which $Y_i \cap \sigma(A) = \Phi$, we see that there exist $e_i \varepsilon H_i$, $\|e_i\| = 1$, (necessarily orthogonal) such that

$$f \varepsilon \langle e_1, e_2, \cdots, e_n \rangle.$$

Furthermore, let $\lambda_i \varepsilon\ Y_i \cap \sigma(A)$.

Then, since $e_i \varepsilon E(Y_i \cap \sigma(A))H$, we have

$$\|(A-\lambda_i)e_i\| < \|(A-\lambda_i) | H_i\| < \varepsilon_i.$$

Now we write A in the form (1).

By part [1], we have

$$\|C\|^2_{HS} < \Sigma^n_{i=1} 2\varepsilon_i^2 < \varepsilon^2.$$

[III] Finally, let $\varepsilon > 0$. Let $\{f_1, f_2, f_3, \cdots\}$ be a spanning set for H. By part [II], there exist $e_1, e_2, \cdots, e_n$ (orthonormal) such that

$$\begin{array}{c} \quad e_1, e_2, \cdots e_n \\ A = \left(\begin{array}{c|c} \begin{matrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & \lambda_n \end{matrix} & 0 \\ \hline 0 & A_1 \end{array}\right) + C_1, \end{array}$$

where $f_1\ \varepsilon\ \langle e_1, e_2 \cdots, e_n \rangle$ and $\|C_1\|^2_{HS} < \varepsilon^2/_2$.

Again by applying part [II] to A_1, there exist $e_{n+1}, e_{n+2}, \cdots, e_{n+m}$ (o.n.) such that

$$\begin{array}{c} \quad e_1, e_2 \cdots e_n \qquad e_{n+1}, e_{n+2} \cdots e_m \\ A = \left(\begin{array}{c|c|c} \begin{matrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & \lambda_n \end{matrix} & 0 & 0 \\ \hline 0 & \begin{matrix} \lambda_{n+1} & & 0 \\ & \lambda_{n+2} & \\ & & \ddots \\ 0 & & \lambda_m \end{matrix} & 0 \\ \hline 0 & 0 & A_2 \end{array}\right) + C_1 + C_2, \end{array}$$

where $f_1,\ f_2\ \varepsilon \langle e_1, e_2, \cdots, e_n, e_{n+1}, e_{n+2}, \cdots, e_m \rangle$.

Continuing in this manner, we get an infinite orthomormal set $\{e_1, e_2, e_3, \cdots\}$ which spans the whole space H, since each f_i belongs to

$\langle e_1, e_2, e_3 \cdots \rangle$.

If we let $C = C_1 + C_2 + C_3 + \cdots$, then the series converges in operator norm, since

$$\|C\|^2_{HS} \ \langle (\varepsilon^2/_2) + (\varepsilon^2/_4) + (\varepsilon^2/_8) + \cdots = \varepsilon^2,$$

and A has the form

$$A = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \ddots \end{pmatrix} + C.$$

Remark. From our construction, $\{\lambda_1, \lambda_2, \lambda_3, \cdots\}$ is contained in $\sigma(A)$, so $\sigma(D) \subset \sigma(A)$.

Theorem 9. Let A be a normal operator on a separable Hilbert space, dim $H = \aleph_0$, then $A =$ a diagonal opeator + a compact operator." [1].

Proof. [7]. By Theorem 7, there is a Hermitiam operator B and a complex valued continuous function g on the spectrum Γ of B such that $A=\int g(t)dE_1(t)$, where $\{E_1(\delta_1):\delta_1\varepsilon\Sigma_1\}$ denotes the spectral measure on the Borel σ-field Σ_1 on Γ, associated with B. The Weyl-von Neumann (Theorem 8 above) says that

B=D+C, where D is a diagonal operator and C is a compact operator.

Also the spectrum Γ_1 of D is contained in the spectrum Γ of B.

By the Weierstrass approximation theorem, there exists a sequence $\{p_n)(\ )\}$ of complex polynomials in a single real variable $x\varepsilon\Gamma$ that converges to $g(\cdot)$ uniformly. Then also $p_n(\cdot|\Gamma_1)\to(\cdot|\Gamma_1)$ uniformly. By the Gelfand-Naimark theorem [4], we see that $p_n(B)\to g(B)$ and $p_n(D)\to g(D)$ in the norm, where $g(B)=\int_\Gamma g(t)dE_1(t)$, by the Gelfand-Naimark theorem, so that $g(B)=A$.

And $g(D)$ can be understood simply as a limit of $\{p_n(D)\}$ which certainly converges, since it is a Cauchy sequence, or $g(D)=\int_{\Gamma_1} g(t)dF(t)$, where F is the spectral measure associated with D. Put $C_n=p_n(B)-p_n(D)$, then each C_n is compact and $\{C_n\}$ converges in the norm to an operator K, which is necessarily compact. We fix a basis of H that makes D diagonal, then $p_n(D)$ are all diagonal operators. If one computes the entries of the matrix of $g(D)$ with the aid of the fact that $p_n(D)\to g(D)$ in norm, we then easily see that $g(D)$ is a diagonal operator as well. But $K=g(B)-g(D)$, so that $A=g(D)+K$. Q.E.D.

Now let us put the diagonal operator $g(D)=W$. Our next goal is to sharpen the above proof of Halmos as follows: We can choose W and K so that $\sigma(W)\subset\sigma(A)$ and $\|K\|$ arbitrary small.

Lemma 10. Let A be a commutative Banach algebra with identity and $x\ \varepsilon\ A$, then $\sigma(x)=\{\varphi(x):\varphi$ is a homomorphism of A onto the complex plane$\}$. [11]

Lemma 11. If B is a maximal commutative subalgebra of the Banach algebra A with identity, and if $x\ \varepsilon\ B$ then

$$\sigma_A(x)=\sigma_B(x)\ [11].$$

Proof. Note that A and B have the common identity. Clearly $\sigma_A(x)\subset\sigma_B(x)$. Now, if $\lambda\notin\sigma_A(x)$, then there is $y\ \varepsilon\ A$ such that $(x-\lambda)y=y(x-\lambda)=1$.

For all $z\ \varepsilon B$, $yz=yz(x-\lambda)y=y(x-\lambda)=zy$. By maximality, $y\ \varepsilon B$. Hence $\lambda\notin\sigma_B(x)$, showing $\sigma_B(x)\subset\sigma_A(x)$. Q.E.D.

The following Theorem looks as a most natural extension of the spectral mapping theorem for polynomials [11].

Theorem 12. Let A be a Banach algebra with identity and x,y εA. Suppose that there is a sequence of polynomials $p_n(\cdot)$ such that $p_n(x)\to y\varepsilon A$ in norm. Then for each $\lambda\varepsilon\sigma(x)$, the numerical sequence $\{p_n(\lambda)\}$ converges and $\sigma(y)=\{\mu:p_n(\lambda)\to\mu;\ \lambda\varepsilon\sigma(x)\}$.

Proof. If B is the maximal abelian subalgebra containing x, then it is also the maximal abelian subalgebra containing y. By the preceeding Lemma 11, we now may assume that A itself commutative. Let $\wedge$ denote the Gelfand mapping of A onto the algebra of all continoous functions on X, where X is the set of all multiplicative linear functionals on A onto the complex plane, equipped with the weak* topology [12]. Let $\lambda \epsilon \sigma(x)$.

Then we take a $\varphi \epsilon X$ such that $\varphi(x)=\lambda$, by Lemma 10. Note that $\hat{\lambda}(\varphi)=\varphi(x)=\lambda$, and

$$\| p_n(x)-y \| = \| p_n(x)-\hat{y} \|_{sup}$$
$$\geq |(p_n(\hat{x})-\hat{y})(\varphi)|, \text{ for each } \varphi\in X$$
$$=|p_n(\lambda)-y(\varphi)|\to 0, \text{ as } n\to\infty$$

Thus

$$p_n(\lambda)\to y(\varphi)=\varphi(y)\epsilon\sigma(y).$$

It follows that $\{p_n(\lambda)\}$ converges and $\{\mu: p_n(\lambda)\to\mu\} \subset\sigma(y)$.

The fact that $\sigma(y)\subset\{\mu: p_n(\lambda)\to\mu\}$ is similarly proved, by noticing that a typical point of $\sigma(y)$ is also of the form $\varphi(y)$ (cf. Lemma 10 and 11). Q.E.D.

The next is a desired improvement of Theorem 9.

Theorem 13. Let A be a normal operator on a separable Hilbert space, dim $H=\aleph_0$, then $A=D+W$, where D is a diagonal operator and W is a compact operator such that

$$\sigma(D) \subset\sigma(A),\ \sigma(W) \subset\sigma(A)$$

and

$\| K \|$ is arbitrary small. [1]

Proof. Let $\mu\epsilon\sigma(W)=\sigma(g(D))$.

By Therorem 12, and the proof of Theorem 9, there is $\lambda\epsilon\sigma(D)\subset\Gamma$

such that

$$p_n(\lambda)\to\mu,\ p_n(\lambda)\to g(\lambda)\epsilon\sigma(g(B))=\sigma(A).$$

Again by Theorem 12, we see that

$$\mu=g(\lambda)\epsilon\sigma(A).$$

Hence

$$\sigma(W)\subset\sigma(A).$$

Now by the proof of Theorem 8, $\| C \|$ $(\leq \| C \|_{HS})$ in the proof of Theorem 9, can be arbitrary small.

A simple modification of the proof of Theorem 9 gives the conclusion. Q.E.D.

References

[1] I.D. Berg, Trans. Amer. Math. Soc. 160 (1971), 365-371

[2] N. Bourbaki, Topologie I générale, Hermann et Cie Act. Sci . Ind. p.363.

[3] Dunford-Schwartz, Linear Operators Part I, Int. Pub. Inc. New York, p.426.

[4] ______________, Linear Operators Part II, p.895 and p.900, 9. Corollary.

[5] ______________, Part II, p.899.

[6] B.R. Gelbaum, Amer. Math. Soc. 15 (1964), 391-392

[7] P.R. Halmos, Continuous Functions of Hermitian Operators, Proc. Amer. Math. Soc. 31 (1972), 130-132

[8] Kuratowski, Topologie II, Warsaw p.23

[9] J. von Neumann, Zur Algebra der Funktional Operationen und Theorie der Normalen Operatoren Math. Ann. 102 (1930), 370-427, p.401 Satz 10.

[10] ______________, Actualites Sci. et Ind, 229 (1935)

[11] Radjavi-Rosenthal, Invariant Subspaces, Springer Verlag, (1973(pp. 4-5

[12] Simmons, Introduction to Topology and Modern Analysis, Mc Graw Hill Book Company, Inc. pp.320-322

[13] B. Sz-Nagy, Spektraldarstelling Linear Transformationen des Hilbertschen Raumes, Springer, Berlin 1942, p.67.

[14] H.Weyl-Rend, Circ. Mat. Palermo 27 (1909), 373-392.