A Note on the Riemann-Stieltjes Integral and the Decomposition Theorem

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Riemann-Stieltjes 積分과 分解定理에 관한 小考

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Summary

In this note, we generalize the definition of the Riemann-Stieltjes integral in [6] and find the some properties using this definition. And also, we investigate the structure of the family of the (K,μ)-pure sets (resp. K-pure) of X where (X,Σ,μ) is a σ-finite measure space, and we can have the Decomposition Theorem from it.

1. The Riemann-Stieltjes Integral

The generalized Remann-Stieltjes integral is based on a definition of the Riemann-Stieltjes integral which depends very explicitly on the order structure of the real line. Accordingly, we begin by discussing integration of the real valued function on intervals.

Definition 1.1. Let [a,b] be given interval. And let f and α be a bounded real valued and monotonic function on [a,b] respectively. For each partition $P=\{x_0, x_1, \ldots, x_n\}$ of [a,b] with $a=x_0<x_1<\ldots<x_n=b$, Let $\Delta\alpha_i=\alpha(x_i)-\alpha(x_{i-1})$. Put

	$\alpha:\nearrow$	$\alpha:\searrow$
$U(P,f,\alpha)$	$\Sigma M_i\Delta\alpha_i$	$\Sigma m_i\Delta\alpha_i$
$L(P,f,\alpha)$	$\Sigma m_i\Delta\alpha_i$	$\Sigma M_i\Delta\alpha_i$

where the notation $\alpha:\nearrow$ and $\alpha:\searrow$ denotes the monotonically increasing and the monotonically decreasing function respectively, and

$$M_i = \sup\,[f(x) : x_{i-1} \leq x \leq x_i]$$
$$m_i = \inf\,[f(x) : x_{i-1} \leq x \leq x_i].$$

And we define

(1) $\overline{\int_a^b} f\, d\alpha = \inf_P U(P,f,\alpha)$

(2) $\underline{\int_a^b} f\, d\alpha = \sup_P L(P,f,\alpha).$

If the left members of (1) and (2) are equal, we denote their common value by

(3) $\int_a^b f\, d\alpha$

or sometimes by

(4) $\int_a^b f(x)\, d\alpha(x).$

The is the generalized Riemann-stieltjes integral of f with respect to α, over [a,b]. If (3) exists, then we say that f is integrable with respect α, and write $f \in \mathscr{R}(\alpha)$. Obviously, the Riemann-Stieltjes integral is a special case of the generalized Riemann-Stieltjes integral since $\alpha:\nearrow$.

In this paper, we will show that the generalized Riemann-Stieltjes integrals are valid where the Riemann-Stieltjes integrals are. It is sufficient to show the case $\alpha:\searrow$ on [a,b].

Proposition 1.2. If P^* is a refinement of P, that is $P^*\supset P$, and $\alpha:\searrow$ on [a,b], then

(5) $L(P,f,\alpha) \leq L(P^*,f,\alpha)$

and

(6) $U(P^*, f, \alpha) < U(P, f, \alpha)$.

Proof. To prove (6), suppose that $P^* = P \cup [x^*]$ is a partition of [a,b] with $a = x_0 < x_1 < \ldots < x_{i-1} < x^* < x_i < \ldots < x_n = b$. Put

$$w_1 = \inf [f(x) : x_{i-1} \leq x \leq x^*]$$
$$w_2 = \inf [f(x) : x^* \leq x \leq x_i].$$

Then $m_i \leq w_1$, $m_i \leq w_2$ where $m_i = \inf [f(x): x_{i-1} \leq x \leq x_i]$. Hence

$$U(P, f, \alpha) - U(P^*, f, \alpha)$$
$$= m_i (\alpha(x_i) - \alpha(x_{i-1})) - (w_1 (\alpha(x^*) - \alpha(x_{i-1}))$$
$$+ w_2 (\alpha(x_i) - \alpha(x^*))$$
$$= (m_i - w_1)(\alpha(x^*) - \alpha(x_{i-1})) + (m_i - w_2)$$
$$(\alpha(x_i) - \alpha(x^*)) \geq 0.$$

If P^* contains k points more than P, we repeat this reasoning k times, and arrive at (6).

The proof of (5) is analogous.

Definition 1.3. Let J be a step function defined by

$$J(x) = \begin{cases} 0 & (x \leq 0) \\ -1 & (x > 0). \end{cases}$$

Proposition 1.4. If $a<s<b$, f is bounded on [a,b], f is continuous at s, and $\alpha(x) = J(x-2)$, then

(7) $$\int_a^b f\, d\alpha = -f(s).$$

Proof. Let $P = \{x_0, x_1, x_2, x_3\}$ be a partition of [a,b] with $a = x_0 < x_1 = s < x_2 < x_3 = b$. Then $U(P,f,\alpha) = -m_2$ and $L(P,f,\alpha) = -M_2$. Since f is continuous at s, so M_2 and m_2 converge to f(s) as $x_2 \to s$. Hence we have (7).

Proposition 1.5. Suppose $c_n \geq 0$ for n = 1, 2, 3, ..., $\sum_{n=1}^{\infty} c_n$ converges, $\{s_n\}$ is a sequence of distinct points in (a,b), and

$$\alpha(x) = \sum_{n=1}^{\infty} c_n J(x - s_n).$$

Let f be continuous on [a,b], then

(8)

$$\int_a^b f\, d\alpha = -\sum_{n=1}^{\infty} c_n f(s_n)$$

Proof. $\alpha(x)$ converges for every x since Σc_n converges. And α:↘ with $\alpha(a) = 0$ and $\alpha(b) = -\sum_{n=1}^{\infty} c_n$. For every $\epsilon>0$, there exists $N>0$ such that $\sum_{n=N+1}^{\infty} c_n < \epsilon$ since α converges. Put

$$\alpha_1(x) = \sum_{n=1}^{N} c_n J(x - s_n), \quad \alpha_2(x) = \sum_{n=N+1}^{\infty} c_n J(x - s_n).$$

Then

$$\int_a^b f\, d\alpha_1 = \sum_{n=1}^{N} c_n \int_a^b f(x)\, dJ(x - s_n)$$

$$= -\sum_{n=1}^{N} c_n f(s_n), \text{ by Proposition 1.5, } \left|\int_a^b f\, d\alpha_2\right|$$

Hence

$$\left|\int_a^b f\, d\alpha = \sum_{n=1}^{N} c_n f(s_n)\right| = \left|\int_a^b f\, d\alpha_2\right| < M\cdot\epsilon$$

$\leq M\, |\alpha_2(b) - \alpha_2(a)| < M\,\epsilon$ where $M = \sup f(x)$.

since $\alpha = \alpha_1 + \alpha_2$. If we let $N \to \infty$, we obtain (8).

Remark 1.6 The integrability of the real valued function with respect to the monotonic function whether α:↗ or α:↘ is invariant, but the value of the integral is not.

2. The Decomposition Theorem

Here we use the concept of pure set by one defined in terms of convex cones in Banach spaces. And we will need to consider sets which are pure for a given B-valued measured relative to a given positive measure, this concept being defined in terms of general convex sets in Banch spaces.

Definition 2.1. Let (X,Σ,μ) be a σ-finite measure space, and let m be a B-valued measure on Σ. If K is a closed convex cone in B (with vertex 0), then a measurable set E is called K-pure for m if $m(F) \in K$ for all $F \subseteq E$. If m is μcontinuous, that is, $m(E) = 0$ whenever $\mu(E) = 0$, $E \in \Sigma$, and if K is any closed convex subset of B, then a measurable set E will be called K-pure for m relative to μ, or (K,μ)-pure, if $A_E(m) \subseteq K$ where

$$A_E(m) = \left\{\frac{m(F)}{\mu(F)} : F \in \Sigma, F \subset E, 0 < \mu(F)\right\}$$

is the average range of m on E.

This is a slightly short proof of **Proposition 2.2** of [4].

Proposition 2.2. Any countable union of (K,μ)-pure sets is a (K,μ)-pure set.

Proof. Let $E = \cup_{i=1}^{\infty} E_i$ where every E_i is (K,μ)-pure and pairwise disjoint. Since $m(F\cap E_i)/\mu(F\cap E_i) \in K$ where $F \subseteq E$ and $o < \mu(F) < \infty$, the convex hull of K is the set

$$A_E(m) = \{\frac{m(F)}{\mu(F)} : F\subset E,\ o<\mu(F)<\infty\}$$

since

$$\frac{m(F)}{\mu(F)} = \Sigma_{i=1}^{n} \frac{m(F\cap E_i)}{\mu(F\cap E_i)} \frac{\mu(F\cap E_i)}{\mu(F)}$$

Since K is convex and closed, $m(F)/\mu(F) \in K$, that is, E is (K,μ)-pure.

Since $m(F) = \Sigma_{n=1}^{\infty} m(F\cap E_i)$ where $m(F\cap E_i) \in K$ for every i and K is closed and convex, so any countable union of K-pure sets is a K-pure set.

Remark 2.3. It is easy to say that the superset of not (K,μ)-pure set is not (K,μ)-pure. The complemnt of (K,μ)-pure set is not (K,μ)-pure if there exists some set which is not (K,μ)-pure. Thus the family of (K,μ)-pure sets of X is not a σ-algebra. We can say about the family of K-pure sets of X similarly.

Definition 2.4. Let f be a measurable function, and let $E \in \Sigma$. Then the essential range of f restricted to E, $er_E(f)$, is defined to be the set of those $b \in B$ such that for every $\varepsilon > o$ the measure of $\{x\in E: \|f(x) - b\| < \varepsilon\}$ is strictly positive.

If $D\subseteq B$, then cl[C.H.D] will denote the closed convex hull of the set D. If is an integrable function, then the indefinite integral of f, m, is the B-valued measure defined by

$$m(E) = \int_E f\, d\mu, \qquad E\in\Sigma$$

Then, we have

Proposition 2.5. If f is an integrable function, and if $E\in\Sigma$ is such that $0<\mu(E)<\infty$, then

$$A_E(m) \subseteqq \text{cl}\ [\text{C.H.}\ (er_E(f))]. \qquad \text{(Cf. [4])}$$

Here we can obtain the relation between the (K,μ)-pure sets and $er_E(f)$. This result gives some motivation for the proof of the Decomposition Theorem.

Proposition 2.6. Let f be an integrable function, and let K be a closed convex subset of B. Then $E\in\Sigma$ is (K,μ)-pure for m if and only if $er_E(f)\subseteq K$.

In the proof of this Proposition, the Proposition 2.5 does an important role. In fact

$$m(F) / \mu(F) \in \text{cl}\{\text{C.H.}\ (er_F(f))\}$$

where $F\subseteq E$ and $0<\mu(F)<\infty$.

Here we can have the Decomposition Theorem.

Theorem 2.7. (Decompositin Theorem). Let (X,Σ,μ) be a σ-finite measure space and let $\overline{A}_E(m)$, the closure of the average range of m on $E\in\Sigma$, be compact where m is μ-continuous. Let (U_i) be a finite collection of open convex subsets of B which covers $\overline{A}_E(m)$ for $1\leqslant i\leqslant n$. Then there exists a finite collection $\{Ei\}$ of measurable sets satisfying the condition $E = \cup_{i=1}^{n} E_i$ and E_i is $(\overline{U}_i,\mu)$-pure for $1\leqslant i\leqslant n$.

Proof. Suppose that $\mu(E) < \infty$. Let

$$a_i = \sup\{\mu(F) : F\subseteqq E,\ F \text{ is } (\overline{U}_i, \mu)\text{-pure}\}$$

for each i, $1\leqslant i\leqslant n$. Let E_{ij} be a sequence of $(\overline{U}_i,\mu)$-pure subsets of E such that $\lim_{j\to\infty} \mu(E_{ij}) = 0$ and $E_i = \cup_{j=1}^{\infty} E_{ij}$. Then E_i is a $(\overline{U}_i,\mu)$-pure set and $\mu(E_i) = a_i$. Let $F = E - \cup_{i=1}^{n} E_i$. Suppose that $\mu(F)>0$ and let b be any extreme point of cl[C.H. (A_F)]. Since $b\in\overline{A}_F$, $b\in U_j$ for some j, that is, $B(b, \delta)\subset U_j$. Then there is a $F'\subseteq F$ such that $\mu(F')>0$ and F is (b,δ)-pure, the closed ball of radius δ at the center $b\in B$. Hence F is $(\overline{U}_j,\eta)$-pure and $F'\cap E_j = \phi$ since $F'\subseteq F$. This contradicts $\mu(E_i) = a_i$. Thus $\mu(F) = 0$. So there exists $\{E_i\}$ of measurable sets with $E = \cup_{i=1}^{n} E_i$ and E_i is $(\overline{U}_i,\mu)$-pure for $1\leqslant i\leqslant n$.

If $\mu(E) = \infty$, then $E = \cup_{i=1}^{\infty} F_i$ where $\mu(F_i)<\infty$ for each i, since μ is σ-finite. We can choose E_{ij} with $E_j = \cup_{i=1}^{n} E_{ij}$ for every j and E_{ij} is $(\overline{U}_i,\mu)$-pure for each i, $1\leqslant i\leqslant n$. Then E_i is $(\overline{U}_i,\mu)$-pure for each i and $E = \cup_{i=1}^{n} E_i$.

References

J. Chang, "On the Range of a Vector valued measure," Jeju Univ. J. 10 (1978), 157-159.

N. Dunford and J.T. Schwartz, "Linear operators," Part, Interscience, New York, 1958.

P.R. Halmos, "Measure theory," Van Nostrand, New York, 1950.

M. Rieffel, "The Radon-Nikodym theorem for the Bochner integral," Trans. Amer. Math. Soc. 131 (1968), 466-487.

W. Rudin, "Functional analysis," McGraw-Hill, New York, 1973.

———, Principles of Mathematical Analysis, McGraw-Hill, New York, 1976.

論 文 抄 錄

本 論文에서는 첫째로 Riemann Stieltjes 積分의 定義를 一般化하여 이 경우에도 積分可能性에는 不變이며 단지 그때의 積分値에 대하여만 相応 変化가 주어진다는 事実을 例를 들어서 보였다. 또한 둘째로 (X, Σ, μ)가 σ-有限 測度 空間일때 이 위에서 定義된 (k, μ)-pure 集合들의 모임의 性質을 調査하고 이로부터 分解 定理를 얻었다.