Some Properties of β- irresolute Maps on Nearly Open Sets

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β - irresolute 寫像에 關한 考察

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Summary

In this paper, we define a β-irresolute map and obtain its characterizations and some properties of the maps on nearly open sets. Moreover, we define a β-Hausdorff space and have its some topological properties and its characterizations.

I. Introduction and Preliminaries

Let T be a topology on a set X, and let "Cl" and "Int" denote closure and interior with respect to T. In 1965, O. Njastad introduced the concepts of nearly open sets as follows; a subset A of a topological space (X,T) is an α-set or β-set if $A \subset Int(Cl(Int(A)))$ or $A \subset Cl(Int(A))$ respectively. We denote the class of all α-sets or β-sets $\alpha(X)$ or $\beta(X)$ respectively. He studied some properties of topological structure using these nearly open sets, and showed that $\alpha(X)$ is a topology but $\beta(X)$ is not a topology and $T \subset \alpha(X) \subset \beta(X)$. And Maheshwari and Thakur developed these theory in 1980. They introduced the concept of α-irresolute map as follows; a map $f:X \to Y$ is said to be α-irresolute if the inverse image of every α-set in Y is an α-set in X. And they studied some properties of β-irresolute maps.

In this paper, we introduce the concept of β-irresolute map and investigate some properties of β-irresolute maps.

II. For β-irresolute maps on nearly open sets

Definition 2.1. A map $f:X \to Y$ is said to be β-irresolute if the inverse image of every β-set of Y is a β-set in X.

The concepts of continuous map, α-irresolute map and β-irresolute map are independent. For,

Example 2.2. (1) Let $f:(R,T) \to (R,L)$ by $f(x) = x$ for all $x \in R$, where T is the usual topology on the real numbers R and L is the lower limit topology on R. Then f is not continuous and not α-irresolute map but f is β-irresolute map.

(2) Let $X = \{a,b,c,d\}$, $Y = \{x,y,z\}$ be equipped with the topologies $T_X = \{\phi, \{a\}, \{b,c\}, \{a,b,c\}, X\}$, $T_Y = \{\phi, \{x\}, Y\}$.

Define $f:X \to Y$ by $f(a) = x$, $f(b) = y$, $f(c) = f(d) =$

z. Then f is continuous but f is not β-irresolute, since $f^{-1}(\{x,y\}) = \{a,b\} \notin \beta(X)$ for β-set $\{x,y\}$. Similarly f is not α-irresolute map.

(3) Let us equip the sets X and Y as (2) with the topologies $T_X = \{\phi, \{a\}, X\}$ and $T_Y = \{\phi, \{x\}, Y\}$ respectively.

Define f: X→Y by f(a) = f(b) = x, f(c) = y, f(d) = z. Then f is β-irresolute and α-irresolute but it is not continuous, since $f^{-1}(\{x\}) = \{a,b\} \notin T_X$ for $\{x\} \in T_Y$.

Proposition 2.3. If f: X→Y and g: Y→Z are β-irresolute maps then g o f: X→Z is a β-irresolute map.

Proof. Let B be a β-set of Z. Then $g^{-1}(B)$ is a β-set of Y, for g is β-irresolute. Therefore $f^{-1}(g^{-1}(B)) = (gof)^{-1}(B)$ is a β-set of X because f is β-irresolute. Hence gof is a β-irresolute map.

Definition 2.4. ([4]) An α-set (β-set) which is closed is termed <u>α-closed</u> (<u>β-closed</u>).

Lemma 2.5. Let $B \subset X_0 \subset X$. If X_0 is closed in X and $B \in \beta(X)$ then $B \subset \beta(X_0)$.

Proof. If B is empty then it is trival. So let B be a nonempty β-set of X. Then $B \subset Cl(Int(B))$. Since B is a nonempty β-set, it is clear that $Int(B) \neq \phi$. Since $B \subset X_0$ and X_0 is closed, we have $Cl(Int(B)) \subset X_0$. Therefore, $Cl(Int(B)) \cap X_0 = Cl_{X_0}(Int(B))$, and $Cl(Int(B)) \subset Cl_{X_0}(Int_{X_0}(B))$. Hence $B \subset Cl_{X_0}(Int_{X_0}(B))$ and $B \in \beta(X_0)$.

We know that the union of β-sets is a β-set but the finite intersection of β-sets is not β-set in general. Say, in (R,T) of Example 2.2 (1), we have $[2,3] \cap [3,4] = \{3\} \notin \beta(R)$ for [2,3] and [3,4] are β-sets.

Lemma 2.6. A subset A of X is an α-set if and only if $A \cap B \in \beta(X)$ for all $B \in \beta(X)$.

Proof. See [1], section 1, proposition 1.

Theorem 2.7. If f: X→Y is a β-irresolute map and A is an α-closed in X, then the restriction $f|_A$: A→Y is a β-irresolute map.

Proof. Since f is β-irresolute, for any β-set V of Y, $f^{-1}(V) \in \beta(X)$. By hypothesis A is closed, hence by lemma 2.5,

$$(f|_A)^{-1}(V) = f^{-1}(V) \cap A \in \beta(A)$$

by lemma 2.6, since A is α-set.

This shows that $f|_A$ is β-irresolute.

Remark. In theorem 2.7, if A is simply closed in X, then $f|_A$ is not always β-irresolute. For if we take A = {b,c,d} and consider example 2.2 (3), then we see that f is β-irresolute but $f|_A$ is not β-irresolute. And if A is not α-closed but β-closed, then for any β-set V of Y,

$$(f|_A)^{-1}(V) = f^{-1}(V) \cap A \notin \beta(A) \text{ by lemma 2.6.}$$

Definition 2.8. ([4]) The complement of an α-set (β-set) is termed a <u>coα-set</u> (<u>coβ-set</u>). We denote the family of all coα-sets (coβ-sets) of X by coα(X) (coβ(X)).

The intersection of all the coα-sets (coβ-sets) containing a set A is termed the α-closure (β-closure) of A. Denote it by $\alpha cl(A)$ ($\beta cl(A)$). Then a set A is coα-set (coβ-set) if and only if $\alpha cl(A) = A$ ($\beta cl(A) = A$).

Lemma 2.9. Let A be a subset of X. Then $x \in \beta cl(A)$ if and only if for any β-set U containing x, $A \cap U \neq \phi$.

Proof. Suppose $x \in \beta cl(A)$. Let U be a β-set containing x such that $U \cap A = \phi$. And so, $A \subset X-U$. But X-U is a coβ-set and hence $\beta cl(A) \subset X-U$. Since $x \notin X-U$, we obtain $x \notin \beta cl(A)$ which is contrary to the hypothesis. Conversely, suppose that every β-set of X containing x meets A. If $x \notin \beta cl(A)$, then there exists a coβ-set F of X such that $A \subset F$ and $x \notin F$. Therefore, $x \in X-F \in \beta(X)$. Hence X-F is a β-set of X containing x but $(X-F) \cap A = \phi$. This is contrary to the hypothesis.

Theorem 2.10. Let f: X→Y be a map. Then the followings are equivalent; (1) f is β-irresolute.

(2) For $x \in X$ and any β-set V of Y containing f(x), there exists $U \in \beta(X)$ such that $x \in U$ and $f(U) \subset V$.

(3) $f(\beta cl(A)) \subset \beta cl(f(A))$ for every $A \subset X$.

(4) $\beta cl(f^{-1}(B)) \subset f^{-1}(\beta cl(B))$ for any $B \subset Y$.

(5) Inverse image of every coβ-set of Y is a coβ-set

of X.

Proof. (1) implies (2): Let $V \in \beta(Y)$ and $f(x) \in V$. Since f is β-irresolute, $f^{-1}(V) \in \beta(X)$ and $x \in f^{-1}(V)$. Put $U = f^{-1}(V)$. Then $x \in U$ and $f(U) \subset V$. (2) implies (3): Let $A \subset X$ and $b \in \beta cl(A)$. We show $f(b) \in \beta cl(f(A))$ by proving each β-set V of Y which contains f(b) intersects f(A). For, finding $U \in \beta(X)$ containing b with $f(U) \subset V$, $b \in \beta cl(A)$ implies that $\phi \neq U \cap A$, which shows $\phi \neq f(U \cap A) \subset f(U) \cap f(A) \subset V \cap f(A)$. (3) implies (4): Let $A = f^{-1}(B)$. Then $f(\beta cl(A)) \subset \beta cl(f(A)) = \beta cl(f(f^{-1}(B))) = \beta cl(B \cap f(X)) \subset \beta cl(B)$, so that $\beta cl(A) \subset f^{-1}(\beta cl(B))$, as required. (4) implies (5): Let $B \subset Y$ be a coβ-set. Then $\beta cl(f^{-1}(B)) \subset f^{-1}(\beta cl(B)) = f^{-1}(B)$, and since always $f^{-1}(B) \subset \beta cl(f^{-1}(B))$, this shows that $f^{-1}(B)$ is a coβ-set. (5) implies (1): This follows from $f^{-1}(Y-B) = X - f^{-1}(B)$ for any β-set B.

III. β-Hausdorff space and β-irresolute maps

Definition 3.1. A space X is said to be β-Hausdorff if for any two distinct points x, y of X, there exist disjoint β-sets U, V of X such that $x \in U$ and $y \in V$.

It is clear that every Hausdorff space is β-Hausdorff.

Proposition 3.2. The following properties are equivalent;

(1) Y is β-Hausdorff.

(2) Let $p \in Y$. For each $p \neq q$, there exists $U \in \beta(Y)$ such that $p \in U$ and $q \notin \beta cl(U)$.

(3) For each $p \in Y$, $\cap \{\beta cl(U): U \text{ is } \beta\text{-set containing } p\} = p$.

(4) The diagonal $\Delta = \{(y,y) : y \in Y\}$ is coβ-set in Y x Y.

We have the following lemma to prove the proposition 3.2.

Lemma 3.3. If $A \in \beta(X)$, and $B \in \beta(Y)$, then $AxB \in \beta(XxY)$.

Proof. $AxB \subset cl_X(Int_X A) \times cl_Y(Int_Y B) = cl_{XxY}(Int_X(A) \times Int_Y(B)) = cl_{XxY}(Int_{XxY}(AxB))$. Conseuqnetly $AxB \in \beta(XxY)$.

Proof of the proposition 3.2: (1) implies (2): Given $q \neq p$, there exist disjoint β-sets U and V containing p and q respectively, which says that $q \notin \beta cl(U)$. (2) implies (3): If $p \neq q$ then there exists β-set U such that $p \in U$ and $q \notin \beta cl(U)$. Hence $q \notin \cap \{\beta cl(U): U \text{ is a } \beta\text{-set containing } p\}$. (3) implies (4): Let $(p,q) \notin \Delta$, then $p \neq q$ and since $p = \cap \{\beta cl(U): U \text{ is a } \beta\text{-set containing } p\}$, there exists some $U \in \beta(Y)$ with $p \in U$ and $q \notin \beta cl(U)$. Since $U \cap \{Y - (\beta cl(U))\} = \phi$, $U \times \{Y - (\beta cl(U))\}$ is a β-set containing (p,q) by lemma 3.3. in $YxY - \Delta$. Hence $YxY - \Delta = \cup [U \times \{Y - \beta cl(U)\}]$ is a β-set. Therefore Δ is a coβ-set. (4) implies (1): If $p \neq q$, then $(p,q) \notin \Delta$. Therefore (p,q) has a β-set UxV of YxY such that $(UxV) \cap \Delta = \phi$. Hence $p \in U \in \beta(Y)$ and $q \in V \in \beta(Y)$ and $U \cap V = \phi$.

Theorem 3.4. If f: $X \to Y$ is a β-irresolute map and Y is β-Hausdorff then G(f) is a coβ-set of XxY.

Proof. Let $(x,y) \in XxY - G(f)$. Then $y \neq f(x)$. Since y is β-Hausdorff, there exist disjoint β-sets W and V of Y such that $f(x) \in W$ and $y \in V$. Moreover, by theorem 2.10 (2), there exist $U \in \beta(X)$ such that $x \in U$ and $f(U) \subset W$, because f is β-irresolute. Therefore we obtain $(x,y) \in UxV \subset XxY - G(f)$. By lemma 3.3, $UxV \in \beta(XxY)$. Hence $XxY - G(f)$ is a union of β-sets of XxY. Therefore $XxY - G(f) \in \beta(XxY)$ since the union of β-sets is a β-set. Consequently, G(f) is a coβ-set of X x Y.

Proposition 3.5. Let X be arbitrary and Y be β-Hausdorff and f: $X \to Y$ be a β-irresolute map and injective. Then X is β-Hausdorff.

Proof. For any $x \neq y \in X$, $f(x) \neq f(y)$ since f is injective. Then there exist disjoint β-sets U, V containing f(x), f(y) respectively. Hence $f^{-1}(U)$, $f^{-1}(V)$ are disjoint β-sets containing x,y respectively. And X is β-Hausdorff.

We recall that a topology is called extremally disconnected if the closure of every open set is open. ([2])

Lemma 3.6. A topology T on X is extremally disconnected if and only if $\beta(X)$ is a topology.

Proof. See [1], section 2.

Proposition 3.7. If f, g: X→Y are β-irresolute maps for extremally disconnected space X and β-Hausdorff space Y, $A = \{x : f(x) = g(x)\}$ is a coβ-set of X.

Proof. Let $y \in X\text{-}A$. Then $f(y) \neq g(y)$. Since Y is β-Hausdorff, there exist disjoint β-sets U, V of Y such that $f(y) \in U$ and $g(y) \in V$. Hence $f^{-1}(U)$ and $g^{-1}(V)$ are β-sets of X because f and g are β-irresolute. Let us put $B = f^{-1}(U) \cap g^{-1}(V)$. Then $y \in B \in \beta(X)$ by lemma 3.6, since X is extremally disconnected. Moreover, $A \cap B = \phi$ for otherwise $U \cap V \neq \phi$. Consequently, $y \in B \subset X\text{-}A$, and hence X-A is a union of β-sets of X, i.e. $X\text{-}A \in \beta(X)$. Therefore A is a coβ-set of X.

Corollary 3.8. If f is a β-irresolute map of a β-Hausdorff space X which is extremally disconnected into itself then the set $A = \{x : f(x) = x\}$ is a coβ-set.

Proof. Let $a \in \beta cl(A)$. If $a \notin A$, then $f(a) \neq a$. Since X is β-Hausdorff, there exist U, $V \in \beta(X)$ such that $f(a) \in U$, $a \in V$ and $U \cap V = \phi$. Since f is β-irresolute, $f^{-1}(U) \in \beta(X)$. Therefore $f^{-1}(U) \cap V$ is a β-set by lemma 3.6, and it contains a. Since $a \in \beta cl(A)$, by lemma 2.9, $f^{-1}(U) \cap V \cap A \neq \phi$. This leads to a contradiction that U and V have a common point. Hence $a \in A$, and $\beta cl(A) \subset A$. Consequently, A is a coβ-set.

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國 文 抄 錄

β - irresolute 寫像에 關한 考察

本 論文에서는, O. Njastad가 定義한 β-set을 利用하여, β-irresolute 寫像을 定義하고 그에 關한 同値條件과 그외의 몇가지 性質을 찾아 硏究하였다.

이 β-irresolute 寫像은 連續寫像과 다르며, 또한 Maheshwari와 Thakur가 定義한 α-irresolute 寫像과도 다른 性質임을 例로써 보였다. 더우기 β-Hausdorff 空間을 定義하여 그의 特性 및 몇가지 位相的 性質을 찾아 證明하였다.