Some Properties of Functions of $\kappa\Phi$—Bounded Variation

李 世 烈*

$\kappa\Phi$—有界變動函數의 性質

Lee Se-youhl

Summary

The space κBV is a banach space. The spaces κBV and κBV is imbeded in $\kappa\Phi BV$. $\kappa\Phi$−bounded variation functions are bounded and if $\lim_{n\to\infty} \frac{1}{n}\Sigma_{i=1}^{n} \phi_i(x) \neq 0$ for $x>0$ then these functions have only simple discontinuities.

Introduction

In convenience we will call a collection $\{I_n\}$ to be the prepartition of $[a, b]$ if (a) every member of $\{I_n\}$ is a closed subinterval of $[a, b]$, (b) $\bigcup\{I_n\} = [a, b]$ and (c) any two members of $\{I_n\}$ are mutually non-overlapping, i.e. that their interiors are disjoint. we will denote $f(I) = f(y) - f(x)$ and $|I| = |y - x|$

* 사범대학 조교수

for $I=[x, y]$.

We consider the supremum of $\Sigma|f(I_n)|$ over all prepartitions $\{I_n\}$ of $[a, b]$ denoted by $\sup\Sigma|f(I_n)|$. A function f is of bounded variation on the closed interval $[a, b]$ if $v_a^b(f) = \sup\Sigma|f(I_n)|<\infty$.

Equivalently we could say a function is of bounded variation on the closed interval $[a, b]$ if there is a positive constant C such that for every prepartitions $\{I_n\}$ of $[a, b]$, $\Sigma|f(I_n)|\leq C$

Cyphert (1982) generalized this idea by considering concave functions κ on $[0, 1]$ in his dissertation. The function κ has the following properties on $[0, 1]$:

κ is continuous with $\kappa(0)=0$ and $\kappa(1)=1$,

(2) κ is concave and strictly increasing and

(3) $\lim_{x\to 0^+} \frac{\kappa(x)}{x}=\infty$

A function f is said to be of κ-bounded variation on $[a, b]$ if there exists a positive constant C such that for every prepartitions $\{I_n\}$ if $[a, b]$

$$\Sigma|f(I_n)|\leq C\Sigma\kappa\left(\frac{|I_n|}{b-a}\right)$$

Note : if $\lim_{x\to 0^+} \frac{\kappa(x)}{x}<\infty$, the set κBV of κ-bounded variation functions is the set BV of bounded variation functions. So, to enlarge the class of functions under consideration the condition (3) has been imposed.

EXAMPLE (1) $\kappa_*(x)=\begin{cases} x(1-\log x), & x\neq 0 \\ 0, & x=0 \end{cases}$

(2) $\kappa_\alpha(x)=x^\alpha$ for $0<\alpha<1$

On the other hand, Schramm (1985) generalized the above idea by considering a sequence $\Phi=\{\phi_n\}$, say Φ-sequence, having the the following;

(1) $\phi_n : [0\ \infty)\to[0\ \infty)$, $\phi_n(0)=0$ and $\phi_n(x)>0$ for $x>0$ and $n\in N$,

(2) ϕ_n are convex for $n\in N$,

(3) $\phi_{n+1}(x)\leq\phi_n(x)$ for all $x>0$, and

(4) $\sum_n\phi_n(x)=\infty$ for all $x>0$.

A function f is said to be of Φ-bounded variation on $[a, b]$ if $V_{\Phi a}^{\ b}(f)=\sup\Sigma\phi_n(|f(I_n)|)<\infty$ where the supremum is taken over all prepartitions $\{I_n\}$ of $[a, b]$. ΦBV denotes the set of all functions on $[a, b]$ such that cf is cf Φ-bounded variation on $[a, b]$ for some $c>0$.

Functions of $\kappa\Phi$-bounded variation

Kim (1986) combined above two concepts [2]

DEFINITION 1. let a real valued function f be defined on the closed interval $[a, b]$. f is said to be of $\kappa\Phi$-bounded variation on $[a, b]$ if there exists a positive constant C such that for every prepartitions I_n of $[a, b]$

$$\Sigma\phi_n(|f(I_n)|)\leq C\Sigma\kappa\left(\frac{|I_n|}{b-a}\right)$$

The total $\kappa\Phi$-variation of f over $[a, b]$ is defined by

$${}_\kappa V_\Phi(f)={}_\kappa V_{\Phi a}^{\ b}(f)=\sup\frac{\Sigma\phi_n(|f(I_n)|)}{\Sigma\kappa\frac{|I_n|}{b-a}}$$

where the supremum is taken over all prepartitions $\{I_n\}$ of $[a, b]$. We denoted by $\kappa\Phi BV$ the collection of all functions f on $[a, b]$ such that cf is of $\kappa\Phi$-bounded variation of $[a, b]$ for some $c>0$. Note : If we take $\phi_n(x)=x$ for all $n\in N$, then $\kappa\Phi BV=\kappa BV$ examined in [1]. If we take $\kappa(x)=x$, the $\kappa\Phi BV=\Phi BV$ examined in [3] Along to forms of κ and Φ−sequence, we have the following easily.

THEOREM 1 (1) For fixed $\Phi=\{\phi_n\}$ and $\kappa_1\leq\kappa_2$, we have $\kappa_1\Phi BV\subset\kappa_2\Phi BV$, and $_{\kappa_2}V_\Phi(f)\leq {}_{\kappa_1}V_\Phi(f)$ if $f\in\kappa_1\Phi BV$.

(2) For fixed κ and $\Phi_1=\{\phi_{1n}\}$, $\Phi_2=\{\phi_{2n}\}$, $\phi_{1n}\geq\phi_{2n}$, we have $\kappa\Phi_1 BV\subset\kappa\Phi_2 BV$, and $_\kappa V_{\Phi_2}(f)\leq {}_\kappa V_{\Phi_1}(f)$ if $f\in\kappa\Phi_1 BV$.

(3) For $\kappa_1\leq\kappa_2$ and $\Phi_1=\{\phi_{1n}\}$, $\Phi_2=\{\phi_{2n}\}$, $\phi_{1n}\geq\phi_{2n}$, we have $\kappa_1\Phi_1 BV\subset\kappa_2\Phi_2 BV$, and $_{\kappa_2}V_{\Phi_2}(f)\leq {}_{\kappa_1}V_{\Phi_1}(f)$ if $f\in\kappa_1\Phi_1 BV$

Since $\kappa(x)\geq x$ on $[0,1]$, we have the following.

COROLLARY 2. $\Phi BV\subset\kappa\Phi BV$, and $_\kappa V_\Phi(f)\leq V_\Phi(f)$ if $f\in\Phi BV$. In particular, $BV\subset\kappa BV$

THEOREM 3. If f is montone, then we have

$_\kappa V_\Phi(f)=\phi_1(|f([a,b])|)$.

Proof. Clearly $_\kappa V_\Phi(f)\geq\phi_1(|f([a,b])|)$. let $\{I_n\}$ be a finite collection of nonoverlapping subintervals of $[a,b]$ and let Σ' denote summation over nonzero terms, then

$$\begin{aligned}\Sigma\phi_n(|f(I_n)|)&=\Sigma'\phi_n(|f(I_n)|)\\&\leq\Sigma'\phi_1(|f(I_n)|)\\&\leq\Sigma'(\phi_1(|f(I_n)|)/|f(I_n)|)|f(I_n)|.\end{aligned}$$

Since ϕ_1 is convex $\phi_1(x)/x$ increases with x, thus the above is not greater than

$$[\phi_1(|f([a,b])|)/|f([a,b])|]\Sigma'|f(I_n)|$$
$$\leq\phi_1(|f([a,b])|)$$

It follows that $V_\Phi(f)\leq\phi_1(f([a,b])$ and so, by Corollay 2, $_\kappa V_\Phi(f)=\phi_1(f([a,b])$

LEMMA 4. If f is $\kappa\Phi$−bounded variation, f is bounded.

Proof. For given partition $a\leq x\leq b$, there is a constant $C>0$ such that

$$\phi_1(|f(x)-f(a)|)+\phi_2(|f(b)-f(x)|)$$
$$<C[\kappa((x-a)/(b-a))+\kappa((b-x)/(b-a))]$$

Thus, $\phi_1(|f(x)-f(a)|)<2C$ so that $|f(x)|<\phi^{-1}_1(2C)+|f(a)|$

LEMMA 5. Suppose that a function f is of $\kappa\Phi$−bounded variation on the closed interval $[a,b]$ and $_\kappa V_\Phi(f:a,b)=C$. Then f is $\kappa\Phi$−bounded variation on each closed interval $[u,v]$ and $_\kappa V_\Phi(f:u,v)\leq 3C$ where $a\leq u<v\leq b$.

Proof. Let $\{I_i\}_{i=1}^n$ be a prepartition of $[u,v]$. then

$$\begin{aligned}&\Sigma_{i=1}^n\phi_i(|f(I_i)|)\\&\leq\Sigma_{i=1}^n\phi_i(|f(I_i)|)+\phi_{n+1}(|f([a,u])|)+\phi_{n+2}(|f([v,b])|)\\&\geq C[\sum_{i=1}^n\kappa\left(\frac{|I_i|}{b-a}\right)+\kappa\left(\frac{u-a}{b-a}\right)+\kappa\left(\frac{b-v}{b-a}\right)]\\&\leq C[\sum_{i=1}^n\kappa\left(\frac{|I_i|}{b-a}\right)+\kappa(1)+\kappa(1)]\\&<3C\sum_{i=1}^n\kappa\left(\frac{|I_i|}{v-u}\right)\end{aligned}$$

Since $\kappa\left(\frac{|I_i|}{b-a}\right)\leq\kappa\left(\frac{|I_i|}{v-u}\right)$ and $\sum_{i=1}^n|I_i|=v-u$ so that

$$\kappa(1)=\kappa\left(\Sigma_{i=1}^n\frac{|I_i|}{v-u}\right)\leq\Sigma_{i=1}^n\kappa\left(\frac{|I_i|}{v-u}\right)$$

If $f\in\kappa BV$ or $f\in\Phi BV$, then f has only simple discontinuities. We have the following.

THEOREM 6. Let f be $\kappa\Phi$−bounded variation on the closed interval $[a,b]$. If $\lim\frac{1}{n}\Sigma_{k=1}^n\phi_k(x)=0$ for $x>0$, then $f(x_0+)$ and $f(y_0-)$ exist for $a\leq x<b$ and $a\leq y<b$ respectively.

Proof. Suppose that $B=\overline{\lim_{x\downarrow x_0}}f(x)>\underline{\lim_{x\downarrow x_0}}f(x)=A$. Then there exists sequences of points $\{x_i'\}_{i=1}^\infty$, $x_i'>x_0$ such that $\lim_{n\to\infty}x_i'=x_0$ and $\lim_{i\to\infty}f(x_i')=A$, and $\{x_j''\}_{j=1}^\infty$ $x_j''>x_0$ such that $\lim_{j\to\infty}x_j''=x_0$ and $\lim_{j\to\infty}f(x_j'')=B$. Thus there exist positive integers N_1 and N_2 such that $f(x_i) \leq A+(B-A)/4$ when $i \geq N_1$ and $f(x''_j) \geq B-(B-A)/4$ when $j \geq N_2$. Now, for each $n = 1, 2 \cdots\cdots$, we can choose points x_k, $k=1, 2, \cdots\cdots, n+1$, alternately from $\{x'_i\}$ and $\{x''_j\}$ so that $x_0 < x_1 < x_2 < \cdots < x_n$ min $(b, x_0+1/n)$ and have $|f(x_{k+1})-f(x_k)| \geq \frac{B-A}{2}$ for $k=1, 2, \cdots\cdots, n$.

Now let ${}_\kappa V_\phi\ (f : a, b)=C$. then for partition $\{x_1, x_2, \cdots, x_{n+1}\}$ of $[a, b]$, we have

$$\Sigma_{k=1}^{n}\phi_k\Big(\frac{B-A}{2}\Big) \leq \Sigma_{k=1}^{n}\phi_k(|f(x_{k+1})-f(x_k)|)$$
$$\leq 3C\Sigma_{k=1}^{n}\kappa\Big(\frac{x_{k+1}-x_k}{x_{n+1}-x_1}\Big)$$
$$=3Cn\kappa\Big(\frac{1}{n}\Big)$$

by lemma 5 and Jensen's inequality, so that

$$0 < \frac{1}{n}\Sigma_{k=1}^{n}\phi_k\Big(\frac{B-A}{2}\Big) \leq 3C\kappa\Big(\frac{1}{n}\Big)$$

letting n go to infinity we contradict the fact $\lim_{n\to\infty}\frac{1}{n}\Sigma_{k=1}^{n}\phi_k\Big(\frac{B-A}{2}\Big) \neq 0$ so that $\lim_{x\downarrow x_0} f(x) = \lim_{x\downarrow x_0} f(x)$ must holds.

The case for left limits is handled in a similar fashion.

Let us consider ${}_\kappa V_\phi\ (Cf)$ as a function of variable C. Since $\Phi=\{\phi_n\}$ is a sequence of convex functions, we have $\phi_n(Cx) \leq C\phi_n(x)$ for $0 \leq C \leq 1$. let ${}_\kappa V_\phi\ (f) < \infty$ and let $0 < C \leq 1$. Then ${}_\kappa V_\phi\ (Cf) \leq C\,{}_\kappa V_\phi(f) \to 0$ as $C \to 0$. With this in mind, we define a norm as follows; let $\kappa\Phi BV_0 = \{f \in \kappa\Phi BV : f(a)=o\}$.

For $f \in \kappa\Phi BV_0$, let $\| f \| = \| f \|_{\kappa\phi} = \inf\ \{k>0 : {}_\kappa V_\phi\ (f/k) \leq 1\}$.

LEMMA 7. (1) ${}_\kappa V_\phi\ (f/\| f \|) \leq 1$

(2) If $\| f \| \leq 1$, then ${}_\kappa V_\phi\ (f) \leq \| f \|$

Proof. (1) Let $k > \| f \|$, then for any prepartition $\{I_n\}$ of $[a, b]$

$$\frac{\Sigma\phi_n(|f(I_n)|/\| f \|)}{\Sigma\kappa(|I_n|/(b-a))} \leq \frac{\Sigma\phi_n(|f(I_n)|/k}{\Sigma\kappa(|I_n|/(b-a))}$$
$$\leq {}_\kappa V_\phi\ (f/k)$$
$$\leq 1$$

Taking supremum over all prepartitions $\{I_n\}$ of $[a, b]$,

${}_\kappa V_\phi(f/\| f \|) \leq 1$

(2) For any prepartition $\{I_n\}$ of $[a, b]$, if $\| f \| \leq 1$,

$$\frac{\Sigma\phi_n(|f(I_n)|)}{\Sigma\kappa(|I_n|/(b-a)} \leq \| f \| \frac{\Sigma\phi_n(f(I_n)/f)\| f \|)}{\Sigma\kappa(|I_n|/(b-a))}$$
$$\leq \| f \|$$

By using this Lemma, we have the following result with the simiar proof of ΦBV_0 (Schramm, 1985).

THEOREM 8. $(\kappa\Phi BV_0, \| \cdot \|)$ is a Benach space

The space $\kappa\Phi BV$ is a Banach space with norm

$$\| f \|_{\kappa\phi} = |f(a)| + \| f-f(a) \|.$$

THEORM 9. Suppose that $\Phi_1=\{\phi_{1n}\}$, $\Phi_2=\{\phi_{2n}\}$ and $\Phi_3=\{\phi_{3n}\}$ satisfy $\phi_{1n}(x)\ \phi_{2n}^{-1}(x) \leq k\phi_{3n}^{-1}(x)$ for all n. Then for all $f \in \kappa\Phi_1 BV_0$ and $g \in \kappa\Phi_2 BV_0$, $fg \in \kappa\Phi_3 BV_0$ and

$$\| fg \|_{\kappa\phi_3} \leq 2k\| f \|_{\kappa\phi_1}\ \ \| g \|_{\kappa\phi_2}$$

Proof. given any $I_n \subset [a, b]$, either $\phi_{1n}(|f(I_n)|) \leq \phi_{2n}(|g(I_n)|)$ or $\phi_{1n}(|f(I_n)|) > \phi_{2n}(|g(I_n)|)$ If $\phi_{1n}(|f(I_n)|) \leq \phi_{2n}(|g(I_n)|)$, then we have the following inequality

$$|f(I_n)g(I_n)/k|$$
$$=\frac{1}{k}\phi_{1n}^{-1}(\phi_{1n}(|f(I_n)|))\phi_{2n}^{-1}(\phi_{2n}(|g(I_n)|))$$
$$<\frac{1}{k}\phi_{1n}^{-1}(\phi_{2n}(|g(I_n)|))\phi_{2n}^{-1}(\phi_{2n}(|g(I_n)|)$$
$$<\frac{1}{k}k\phi_{3n}^{-1}(\phi_{2n}(|g((I_n)|))$$
$$=\phi_{3n}^{-1}(\phi_{2n}(|g(I_n)|))$$

Thus, $\phi_{3n}(|f(I_n)g(I_n)|/k) \leq \phi_{2n}(|g(I_n)|)$

If $\phi_{1n}(|f(I_n)|) > \phi_{2n}(|g(I_n)|)$, then a similar argument shows that

$$\phi_{3n}(|f(I_n)g(I_n)|/k) \leq \phi_{1n}(|f(I_n)|)$$

Therefore we have

$$\Sigma\phi_{3n}(|f(I_n)g(I_n)|/k)/\Sigma\kappa(|I_n|/b-a)$$
$$\subset[\Sigma\phi_{1n}(f(I_n)|)/\Sigma\kappa(I_n|/b-a)]$$
$$+[\Sigma\phi_{2n}(|g(I_n)|)/\Sigma\kappa(|I_n|/b-a)]$$

Thus, $fg \in \kappa\Phi_3 BV$

Let $\varepsilon > 0$. Without loss of generality, assume $\| f \|_{\kappa\phi_1} = \| g \|_{\kappa\phi_2} = 1$ By the convexity of ϕ_{3n}, we have

$$\Sigma\phi_{3n}(|f(I_n)g(I_n)|/2k(1+\varepsilon)^2)/\Sigma\kappa(|I_n|/b-a)$$
$$=\tfrac{1}{2}\Sigma\phi_{3n}([|f(I_n)|/1+\varepsilon][/g(I_n)|/1+\varepsilon]/k/\Sigma\kappa(|I_n|/b-a)$$
$$\leq\tfrac{1}{2}\Sigma\phi_{1n}(|f(I_n)|/1+\varepsilon)/\Sigma\kappa(|I_n|/b-a)$$
$$\leq\tfrac{1}{2}+\tfrac{1}{2}=1$$

Thus $_{\kappa}V_{\phi_3}(fg/2k(1+\varepsilon)^2)=1$, $\| fg \|_{\kappa\phi_3} = 2k(1+\varepsilon)^2$ and the theorem follows by letting $\varepsilon \to 0$.

References

Cyphert, D. S. 1982, ***Generalized functions of bounded variation and their application to the theory of harmonic functions***, dissertation, Vanderbilt Univ., Tennessee, USA.

Kim, S. K. and Kim, J. S. 1986. ***Functions of*** $\kappa\Phi$—bounded variation, Bull. Korean Math. Soc. 23, 7-175.

Schramm, M. 1985, ***Functions of Φ—bounded variation and Rieemam—Stieltjes integration***, Trans. Amer. Math. Soc. 287, 49-63.

〈摘要〉

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$\kappa\Phi$BV은 바나크공간이다. ΦBV과 ΦBV은 $\kappa\Phi$BV에 매몰된다. $\kappa\Phi$—유계변동 함수들은 유계이고 만일 $x>0$에 대하여 $\lim_{n\to\infty}\frac{1}{n}\Sigma_{i=1}^{n}\ \phi_i(x)\neq 0$이면 이 함수들은 단순불연속점만을 갖는다.