Normal - Theory Prediction Intervals; Robustness and Alternatives in the Simulation Context

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Table of Contents

I. Introduction

1. Motivation for a Study on Prediction Intervals.

Computer simulation is a general, powerful, and widely used technique for addressing problems in diverse areas of management, operations, and engineering. The power and flexibility of simulation stem from its ability to "try out" alternatives in a "harmless" way by experimenting with the system itself.

All management - science methods make use of mathematical models, so share this power with simulation. However, the class of models that can be evaluated by analytical, non - simulation methods is narrower than that amenable to simulation : thus, use of non - simulation methods requires more restrictive assumptions to form the model, possibly harming the validity of the model and the results. Simulation allows us to study models that are more complex, thus being more valid.

Traditionally, the statistical tools for analyzing the output from computer simulation

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have been confidence intervals (c.i.'s) for expected performance. As an important example, in a manufacturing simulation a c.i. might try to bracket the expected daily production (the average production over infinitely many days). This is appropriate if operation is static and we are interested in average system behavior far into the future. However the c.i. goal may be inappropriate if the facility is flexible, operating in a fixed configuration only briefly, so that the long-run average daily production will never even be attained, or even approached.

As an alternative, a prediction interval (p.i.) could be formed, which instead brackets the next day's production, or (more generally) the average of a fixed small number of future days' production. Despite their apparent appeal, p.i.'s have not been investigated in simulation. In particular, their standard development assumes normality, which is probably never true in realistic simulations. This study will examine p.i. robustness, specifically in the simulation context, to provide guidelines on their use.

2. Statistical Intervals.

In general, there is a need for multiple replications of a simulation in order to effect a valid analysis. If we let X_j be the output measure from the jth entire simulation, then carrying out n replications results in the data $X_1, X_2, \cdots, X_n$. To make the discussion more concrete, suppose that X_j represents the production during the jth simulated shift of a manufacturing system, and so the simulation output $X_1, X_2, \cdots, X_n$ then gives the production across n identical but independent simulated shifts.

The above output data can be regarded as a sample of n observations drawn from a population with a certain distribution. Let's denote its density function and distribution function by $f(x; \theta)$ and $F(x; \theta)$ respectively. Using this sample, we can construct a statistical interval (L_1, L_2) to infer about the population or the next sample. Statistical intervals can be classified into three frequently used categories : confidence intervals, tolerance intervals, and prediction intervals [8]. There are two types of tolerance intervals : β-content and β-expectation [4].

Their definitions are as follows.

A confidence interval is an interval which contains a population parameter θ with a specified probability γ. That is (L_1, X_2) is defined by

$$\Pr(\ \theta \in (L_{1.}\ L_2)) = \gamma, \qquad (1)$$

where θ represents, for example, the mean, standard deviation, median, mode, or a shape parameter of the population distribution.

A β -content tolerance interval is an interval which can be claimed to contain at least a certain portion β of the population distribution with a specified probability γ.

$$\Pr(\int_{L_1}^{L_2} f(x;\theta)dx \geq \beta) \ = \ \Pr(\ F(L_2;\theta) - F(L_1;\theta) \geq \beta\ ) \ = \ \gamma. \qquad (2)$$

A β-expectation tolerance interval is an interval which on average contains a certain portion of the population distribution.

$$E[\ F(L_2;\theta) - F(L_1;\theta)\ ] = \beta. \qquad (3)$$

A prediction interval is an interval which contains (a) statistic(s) of the next sample drawn from the same population with a specified probability.

$$\Pr\{\ \bigcup_i\ [t_i \in (L_1, L_2)]\ \} = \gamma, \qquad (4)$$

where t_i represents a statistic of the next sample. A variety of sample statistics may of interest. The following is a list of examples of t_i which are intended to be contained in a prediction interval.

The next observation y_1,

At least k out of m next observations,

The mean of the next sample of size m,

The smallest observation in the next sample, $y_{(1)}$,

The largest observation in the next sample, $y_{(m)}$,

All the means of the next r samples, etc.

Here, y_i refers to an observation in the next sample we want to predict on. Notice different notations x_i and y_i are used to distinguish between the observations used to construct a p.i. and those about which a prediction is to be made; both of them are drawn from the same population.

One- and two-sided simultaneous p.i.'s to contain the values of each m future observations from a normal population are reviewed by Hahn [8]. The methods are discussed in greater detail in two further articles [6 and 7] and some approximate procedures are given by Chew [1]. The special case of a single future observation, i.e., $k = 1$, was also discussed by Hahn [5].

Two-sided prediction interval to contain with probability r the next observation from the same exponential distribution has lower and upper limits $\bar{x}/F[2n, 2; (1+r)/2]$ and $\bar{x}/F[2, 2n; (1+r)/2]$, where $F[2n, 2; (1+r)/2]$, is the $100(1+r)/2$'th percentile of the F distribution with (2n, 2) degrees of freedom [9]. Note that the degrees are reversed in the lower and upper limits. The p.i.'s for a sample from an exponential distribution can be easily extended to provide p.i.'s for a sample from a gamma distribution with a known value of the shape parameter.

A table for calculating a p.i. for the next observation from the same Weibull distribution is presented by Fertig [3]. This tabulation is obtained from 20,000 Monte Carlo simulations, and contains a statistic which can be used to form a p.i. with confidence level 95% from a sample of size $n = 5(5)25$.

Prediction intervals for the normal distribution may be used for the log-normal distribution. One must then work with the logarithms of the log-normal data and convert the resulting limits of the p.i. from the logarithmic scale back to the original data scale.

Distribution-free p.i.'s are based upon methods developed by Wilks [13]. One and two-sided distribution-free p.i.'s which are based on the smallest and/or largest ordered observations are derived by Hall [10]. Tables which give the probabilities associated with various sample sizes are presented.

In this study we limited our experiments to two-sided prediction intervals for the next observation due to cost restrictions Other types of p.i.'s can be a subject of a further studies.

A prediction interval for the next observation can be defined by

$$\Pr\{\, y_1 \in (L_1, L_2) \,\} = \gamma \tag{5}$$

which is simply (3) with β replaced by γ. That is, the p.i. for the next observation with confidence level γ is the same as the γ- expectation tolerance interval.

Ⅱ. Definitions and Formulas

1. Measure of the Actual Coverage of a P.I.

1) Analytical measure

Once a p.i. (L_1, L_2) is obtained, we can measure its actual coverage by the formula,

Coverage $= F(L_2;\ \theta) - F(L_1;\ \theta)$,

if the value $F(x,\ \theta)$ can be calculated for given x.

2) Empirical measure

It is impossible or very expensive to calculate $F(x,\ \theta)$ for some distributions. In this case we can measure the actual coverage of a p.i. by checking if it really contains the next observation. The actual coverage will be measured as 1 if the p.i. actually covers the next observation, and 0 otherwise. This measure will yield an unbiased estimate of the true actual coverage if it is averaged over a large number of independent p.i.'s constructed by the same procedure.

It cannot be regarded as economical to use just one observation to measure the actual coverage of a p.i. which might have been constructed consuming a fairly large number of observations through a complicated procedure requiring a heavy computational effort, because the resulting measure would be 0 or 1 which may be far from the true value. Instead we can use m observations for the measurement Then the empirical measure of the actual coverage can be defined as

$$\text{Coverage} = (1/m) \sum_{i=1}^{m} I_{[y_i \in (L_1,\ L_2)]}$$

If m is large, this produces an accurate measure of the actual coverage of a particular interval, but consumes observations which might be used to construct another prediction interval. To be specific, suppose we are given a budget which allows us only n observations. Then our experiment should be designed under the constraint

$$(n + m)\ r \geq N$$

where

n = the number of observations used to construet a p.i.,

m = the number of observations used to measure the actual coverage, and

r = the number of p.i.'s constructed and measured.

Put in words, given n, we can afford fewer p.i.'s if we want to measure the performance of individual p.i.'s more exactly. Therefore a trade off is inevitable between m and r.

An alternative is to determine $m = n$ to the effect "Spend more on the measure if more is invested in the construction of a p.i." But this alternative is not appropriate when we want to examine the effect of the sample size on the coverage performance of a p.i. construction procedure because the variation of the measured actual coverage must depend on the number of observations used for measurement as well as the characteristics of a p.i. Considering this point, we chose m = 30 regardless of the sample size, type of population distribution, and fitted distributions. This choice is quite arbitrary, but consideration is given to the fact that the most frequently used sample size is in the range of 20 through 40.

2. Length of the Interval

The interval length is $L_2 - L_1$. Half length is not addressed in this study because there is no symmetry in the interval except in the case of fitting a normal distribution to the data. We made a modification as Length = $L_2 - \max(0, L_1)$ in the case of a population distribution with a nonnegative support. It is a reasonable assumption that, with a decent sample size, we can recognize that the support is nonnegative when it actually is. So, we can cut off the negative part of a p.i. even if a procedure renders a p.i. which includes the negative part by chance.

3. Coverage Functions of P. I. 's

A study on the robustness of p. i. 's can be executed by measuring the frequency that independent p. i. 's cover the next observation at a particular confidence level. Evaluation is made by comparing the coverage frequency tables presented for a few commonly used confidence levels. This becomes troublesome when we are interested in a confidence level which is not shown in the tables. As a way to go around this defect, the concept of coverage function introduced first by Schruben [12] is adopted to evaluate prediction intervals in the whole range of confidence levels.

Suppose a p. i. is constructed by a procedure that specifies an interval, $(L_1, L_2) = I(\gamma, X)$, as a function of a confidence level X and the data X. The procedure is based on assumptions about the random characteristics of the data. In situations where these assumptions hold, the procedure will yield an interval that contains the next observation y_l with a probability γ, i. e.,

$$\Pr\{y_1 \in I(\gamma, X)\} = \gamma \qquad (6)$$

The interval, being a function of data, is random.

The effects of incorrect assumptions can be graphically illustrated and statistically tested by considering a random variable γ^* defined for the procedure as

$$\gamma^* = \inf\{\gamma \in [0,\ 1];\ y_1 \in I(\gamma,\ X)\}. \qquad (7)$$

That is, γ^*, is the smallest confidence level such that the next observation y_1 is contained in the interval $I(\gamma,\ X)$. When the interval estimation procedure is based on true assumptions, γ^* will be uniformly distributed;

$$\Pr\{\gamma^* \le \gamma\} = \Pr\{y_1 \in I(\gamma, X)\} = \gamma. \qquad (8)$$

The first equality in (8) depends on

$$y_1 \ \varepsilon\ I(\gamma, X) \subset I(\gamma, X) \text{ for all } \gamma \ge \gamma^*$$

requiring the interval to be nested. The second equality is the definition (6). The distribution function of γ^*,

$$F_\gamma(\gamma) = \Pr\{\gamma^* \le \gamma\}$$

is called the coverage function. It can be interpreted as the probability that a p.i. actually contains the next observation when it is constructed for a confidence level γ.

In an experiment where r p.i.'s are tested, the coverage function can be measured by the empirical distribution of γ^* denoted as

$$G_{\gamma^*}(\gamma) = (1/r)\sum_{j=1}^{r} I_{[\gamma_j^* \le \gamma]}$$

Here, γ_i^*, $i = 1, \cdots, r$ are the observed values of the random variable γ^* for a set of r prediction intervals and IE is the indicator function for event E. The choice of a particular confidence level can be avoided. Instead comparison of the empirical coverage functions of several p.i. construction procedures provides us with information on which procedure is the most advantageous in a particular sampling situation.

4. Formulas to Determine the Intervals

Suppose we have a sample $\{x_1, x_2, \cdots x_n\}$ obtained from a population of an unknown distribution form and distribution parameter values. Further suppose we want to construct a two-sided p.i. $I(\gamma, X) = (L_1, L_2)$ so that both sides out of the interval take the same probability $(1-\gamma)/2$. Then the formulas for L_1 and L_2 can be presented as follows depending on the fitted distributions.

1) Fitting Normal Distributions

$$L_1 = \overline{X} - t(n-1, (1+\gamma)/2) * S\sqrt{1+1/n}, \qquad (9)$$

$$L_1 = \overline{X} + t(n-1, (1+\gamma)/2) * S\sqrt{1+1/n}, \qquad (10)$$

where, $\overline{X}$ = sample mean, and S = sample standard deviation.
That is, $\overline{X} = \hat{\mu}$ and $S^2 = \hat{\sigma}^2$ based on the assumption $X_i \sim i.i.d.\ N(\mu, \sigma^2)$.

2) Fitting Gamma Distributions

$L_1 = F^{-1}((1-\gamma)/2;\ \hat{\alpha},\ \hat{\beta})$

$L_l = F^{-1}((1+r)/2;\ \hat{\alpha},\ \hat{\beta})$

where $F^{-1}((x,\ \hat{\alpha},\ \hat{\beta})$ is the inverse distribution function of the Gamma distribution with $\hat{\alpha}$ and $\hat{\beta}$. Also, $\hat{\alpha}$ and $\hat{\beta}$ are estimates for α and β respectively. (Refer to Section 5.2.2 of the reference [11]and [2])

The Gamma distribution has a nonnegetive support. So, we may have trouble if we want to fit a Gamma distribution to data drawn from a normal distribution, whose support includes negative values. We treated the data with negative values as if they are 10^{-50} a value positive but very close to 0. Also, we avoided having to test for the population distribution which may generate a sample with many negative observations.

It should be noted that the inverse distribution function of the Gamma distribution whose parameter values are equal to the estimates is used. This is an approximation. Since an estimate, being a function of data, is a random variable, the resulting interval is random. It is expected that the variation in the estimates for the parameters exert a downward effect on the actual coverage when the population distribution has tapering tails on both sides. The reason is that the loss in the actual coverage caused by overly narrow p.i.'s cannot be properly compensated by the gain resulted by overly wide p.i.'s because of the tapering shape of the tails. Since the variation in the estimates would be greater when smaller samples are used, some kind of adjustment should be made on the boundaries so as to expand the interval to a degree depending on the sample size. But no theory has been developed in this area yet, making inevitable the use of the approximation formulas presented above.

3) Fitting Weibull Distributions

$L_l = F^{-1}((1-r)/2;\ \hat{\alpha},\ \hat{\beta})$

$L_l = F^{-1}((1-r)/2;\ \hat{\alpha},\ \hat{\beta})$

where $F^{-1}((x,\ \hat{\alpha},\ \hat{\beta})$ is the inverse distribution function of a Weibull distribution with $\hat{\alpha}$ and $\hat{\beta}$. Also, $\hat{\alpha}$ and $\hat{\beta}$ are are estimates for α and β. (Refer to Section 5.2.2 of the reference [11]).

The same comments can be made here as those addressed in the case of fitting

Gamma distributions. That is, the above formulas are approximations. A compromise is made on the negative values in the data as in the case of fitting Gamma distributions.

4) Fitting Log-normal Distributions

We can take a logarithmic transformation of the data, apply the same formulas (9) and (10) to the transformed data. Then a back transformation (i.e., exponentiation) is applied to the boundaries obtained at the above stage. Negative values'in the data are treated as a very small positive value.

5) Distribution-free prediction Intervals

Suppose we have sorted the original data $x_\cdot$, $\cdots$, x_n of a continuous random variable X to get $x_{(1)}$, $\cdots$, $x_{(n)}$ in an ascending order. Then it is known that

$$\begin{aligned} \Pr(X \leq x_{(1)} \} &= \Pr\{X \geq x_{(n)}) = 1/(n+1) \\ &= \Pr\{ x_{(j_1)} \leq X \leq x_{(j)} \} \quad \text{for } j = 1, \cdots, n. \end{aligned}$$

These equalities imply that $(x_{(1)}, x_{(n)})$ can be used as the prediction interval with confidence level r = 1-2/(n+1. In general, $(x_{(i)}, x_{(n-i+1)})$ makes a p.i. with r = 1-2i/(n+1), i=1, $\cdots$, [n/2]. As sample size increases we can get a p.i. with a confidence level close enough to the value of interest.

If we want to stick to a prespecified confidence level r and can not be satisfied with the closeness, an approximation is inevitable to determine the interval. A simple and straightforward approximation is given as follows, based on the assumption that the population distribution can be well approximated by uniform density functions within each of the subintervals $(x_{(i)}, x_{(i+1)})$, i = 2, $\cdots$, n-1 :

For n given and r specified, solve the equation,

$r = 1-2i/(n+1)$

for i. Suppose the solution i^* is expressed as

$i^* = i_0+\delta$

where $i_0 = [i^*] > 1$, and $0 < \delta < 1$. Then

$x_{(i_0)} \leq L_1 < x_{(i_0+1)}$.

The proposed approximation formula for the lower boundary is

$$L_1 = x_{(i_0)} + \delta(x_{(i_0+1)} - x_{(i_0)})$$

Similarly,

$$L_2 = x_{(n-i_0+1)} - \delta(x_{(n-i_0+1)} - x_{(n-i_0)})$$

The above formulas are applicable only when $i_0 > 1$ (i.e. $1-2/(n+1) > r$). $i_0 = 0$ implies $L_1 < x_{(1)}$ and $x_{(n)} < L_2$. If we pretend a complete ignorance about the shape of the population distribution, there's no reasonable way to determine the interval. Any method to set the boundaries outside the range of the data is subject to argument. We excluded this problem from the scope of this study.

Ⅲ. Experimental Design

1. Population Distributions

Several continuous parametric distributions with various parameter values as well as two simulation models are chosen to draw samples from. Their list is as follows :

1) Normal distributions
2) Gamma distributions
3) Weibull distributions
4) Log-normal distributions

We confined the experiment to the cases where the scale parameter has value 1 because the effects of the scale parameter value is known. The part $(x-\mu)^2/\sigma^2 = (x/\sigma-\mu/\sigma)^2$ of the normal density function implies that σ is a kind of scale parameter. So, we set $\sigma=1$ for the same reason mentioned above. Also, we set the value of the location parameters of the Gamma, Weibull, or log-normal distributions to be zero, assuming its estimation doesn't offer a problem.

5) Average delay time of the first n customers in a M/M/1 queue with traffic intensity t which starts empty and idle, with $t=0.9$ and $n=25$.

6) A reliability model consisting of three components which will function as long as component 1 works and either component 2 or 3 works. If G is the time to failure of the whole system and G_i is the time to failure of components $i(i =$

1, 2, 3), then $G=\min(G_1, \max(G_2, G_3))$. It is assumed that the G's are independent random variables and that each G_i has a Weibull distribution with $\alpha=0.5$ and $\beta=1.0$.

2. Fitted Distributions

As suggested in Section 2.4, five p.i. construction procedures are evaluated. Four procedures are based on fitting to data the following distributions

Normal distributions,

Gamma distributions,

Weibull distributions, and

Log-normal distributions.

Also, distribution-free p.i.'s are evaluated. The location parameter value is assumed to be zero when we fit a Gamma, Weibull, or log- normal distribution to the data.

3. Performance Measures

Actual coverage and interval length are chosen as performance variables. Choice of actual coverage is a matter of course. Interval length is chosen since smaller and less variable interval length is a desirable characteristic of a p.i. because it means more exact information on the location of the next observation. Two summary statistics are used : mean and standard deviation. For a comparison in tabular presentations, the performance is measured for the p.i.'s with the confidence level of 0.70, 0.80, 0.90, 0.95, and 0.99.

4. Use of Random Nwnber Stream

The prime modulus multiplicative linear congruential generator

$Z_i = (7^5 * Z_{i-1}) \pmod{2^{31}-1}$

based on Schrage's portable random number generator RAND is used.

IV. Interpretation of the Experimental Result

1. Robustness of the P. I.'s Constructed on the Normality Assumption

Gamma and Weibull distributions with location parameter value 0 are used as population distributions to analyze the coverage performance of the p. i.'s constructed on the normality assumption (referred to as normal p. i.'s hereafter) in situations where the assumption is erroneous. The gamma distribution converges to a normal distribution as the value of the shape parameter α increases, while it becomes seriously skewed as α approaches zero. So, we can test the performance of normal p. i.'s in diverse situations which cover both a good satisfaction and a serious violation of the normality assumption by using Gamma distributions. Weibull distributions are added to increase the variety of tested population distributions. Even though Weibull distributions don't have the property mentioned for Gamma distributions, they include various shapes. Some are pretty close to normal, others are extremely skewed.

It is expected that a false assumption causes the actual coverage to deviate from the designed level. The degree and direction of the deviation is analyzed for a few values of the confidence level. Also, the effect of sample size used to construct a p. i. is examined.

Findings are presented by population distribution forms.

1) Gamma Distributions

As shown in Table 1 in the appendix, the performance of all the types of p. i.'s is good for α greater than 9.0 in the sense that the absolute value of the gap between the actual coverage of a prediction interval and the desired coverage (referred to as coverage gap hereafter) is trivial.

Let's take a closer look into the case where α = 50.0. The Gamma distribution with α = 50.0 is identical with the distribution of the sum of 50 observations independently drawn from an exponential distribution, and would be quite similar to a normal distribution by the Central Limit Theorem. The coverage shown in the right column (corresponding to infinite sample size) shows the degree of similarity. (Notice that the actual coverage for the case of infinite sample size is analytically calculated by fitting a normal distribution to the population distribution so that both have the same mean and variance.) The coverage gap is less than 0.25% for all the presented confidence levels, implying that both distributions are practically the same. So, we can expect the coverage performance of p.i.'s would be excellent regardless of the sample size, since $X_i \sim N_i(\mu, \sigma^2)$ implies $(\overline{X}-\mu)/\hat{\sigma} \sim t_{n-1}$, validating the theoretic basis for the construction procedure of normal p.i.'s.

We know a Gamma distribution gets deformed from the shape of a normal distribution as α decreases. To get information on how fast this deformation develops, let's examine the part of the right column corresponding to the case where α = 9.0. We can read that the coverage gap has somewhat increased. It is hard to talk about the seriousness of the gap. It should be determined depending on the strictness with which we want to stick to the specified confidence level. 1% gap can be regarded all right for the confidence level 70%, while 0.5% gap may be considered too bad for the confidence level 99%. Anyway the deformation from the shape of a normal distribution is not so serious that we can expect almost the same performance of p.i.'s corresponding to different sample sizes.

As α continues to decrease, the deviation from normality gets more serious. If α comes to be 1.0, the Gamma distribution becomes identical to an exponential distribution, stops having two tails, and has the mode at 0. If α is less than 1, more of the distribution falls on a region close to zero and becomes extremely skewed.

The case of a Gamma distribution with α = 0.01 may have no practical value by itself because even a small sample will reveal the nonnormality of the population distribution and thus prevent us from applying the normality assumption in such a case. But this distribution is taken here to study what performance pattern normal p.i.'s show in extreme situations An interesting finding is the fact that the actual coverage is very insensitive to the change in the designed coverage, but sensitive to the difference in the sample size. The actual coverage changes from 98.3%

to 99.01% for the sample of infinite size and from 77.4% to 78.2% for the sample size 5 while the designed coverage ranges from 70% to 99%(Table 1). This observation can be interpreted as follows : Since α is very small, most of the probability falls in the interval [0. α]. This interval is included even in a p.i. with a low level of confidence because the standard deviation (which is the same as the square root of α) is 10 times larger than the mean. That is why the actual coverage of 70% p.i. is enormously high. But an increase in the designed coverage causes a trivial increase in the actual coverage because the only difference between a p.i. with high confidence level and one with a low confidence level is the length of the included tail which contributes little probability.

The observations we have made thus far can be well summarized and enriched by the Graph 1 and Graph 2 in the appendix. These graphs visualize the conformity to a normal distribution of Gamma distributions by comparing the actual and desired coverage of normal p.i.'s constructed using the perfect information on the distribution parameters. Graph 1 shows that the normality assumption may not hurt so much for the Gamma distributions with α greater than 9 but may hurt for those with $\alpha < 4$ especially for a certain range of confidence level. Graph 2 reinforces our observations regarding the Gamma distributions with $\alpha < 1$. Because the distributions have the mode at 0, there is a point of confidence level where the actual coverage suddenly stops a fast increase. The magnitude of the coverage gap and the range of confidence level corresponding to a serious coverage gap becomes greater and wider as α approaches 0. For all α values there is a range of confidence level close to 100% where overestimation of coverage happens.

2) Weibull Distributions

As Table 2 shows, normal p.i.'s perform quite well for Weibull distributions with $\alpha > 2$. But, in case of $\alpha < 1$, they show a similar performance to that observed in case of Gamma aistributions. That is, their performance degenerates as α value decreases past 1 in a similar way to that observed in the case of Gamma distributions. One of the interesting observations is the fact that the coverage performance does not tend to become perfect as the value of α increases, in contrary to the case of Gamma distributions. Actually, the conformity to a normal distribution is better in case of Weibull with $\alpha = 2$ than, for example, in case of $\alpha = 50$, as can be identified in Table 2.

3) General Comments on the Findings

We can draw the following conclusions from the observations we have made thus far.

Normal p.i.'s are not robust to the violation of the normality assumption. The fact that normal p.i.'s work well for the Gamma distributions with $\alpha > 9$ and the Weibull distributions with $\alpha > 2$ doesn't advocate for the robustness. The observed good performance is caused by the fact that the population distributions themselves are close to a normal distribution, not by the characteristics of normal p.i.'s. Actually, normal distributions have only one shape, and thus lack the flexibility to fit diverse shapes of distributions.

We can guess how the performance of normal p.i.'s would change as the value of the shape parameter changes in the cases of some distribution types such as Gamma distributions. But there is no critical value of the shape parameter where we should stop applying the normality assumption because the degeneration in the coverage performance develops gradually as α changes. What is worse is the fact that the coverage performance differs for different level of confidence. The coverage gap is slight for some ranges of confidence level, serious for others.

In this context, it must result in impracticable tabulations subject to arguments if we try to tabulate the combinations of population distribution type, shape parameter value, range of confidence level, and the sample size with which normal p.i.'s work well. Instead it would be worthwhile to find a substitute applicable to more diverse sampling situations.

2. Evaluation of Different P.I. Construction Procedures

Five p.i. construction procedures are applied to various population distribution types to reveal the comparative merits and defects of each procedure. Rather unusual names are used for the simplicity of description : normal p.i.'s, Gamma p.i.'s, Weibull p.i.'s, log-normal p.i.'s, and distribution-free p.i.'s. Here, for example, Gamma p.i.'s refer to the p.i.'s constructed by fitting a Gamma distribution to the data, and so on.

Because the main purpose is to identify relatively advantageous p.i. construction procedures, the experimental result is presented by the tables and the graphs which simultaneously show the performance of all the procedures to facilitate the comparison among procedures. The value of the performance variables are set to be 0.0 when they could not be measured for some reason.

1) Normal Distributions

The cases with $\mu > 3$ are not used because of the considerable probability that a sample would contain many negative values, which is troublesome for Weibull and log-normal p.i.'s. Gamma p.i.'s are not tested here due to the limits on the calculation of the inverse distribution function of Gamma distributions.

If samples of the same size are used, the actual coverage and interval length of normal p.i.'s and distribution-free p.i.'s are independent of the value of μ (Table 3). This is not surprising since different value of μ means different locations of the distribution. Normal p.i.'s and distribution-free p.i.'s can move along with a change in the location of the population distribution. But the value of μ exerts an effect on the actual coverage of the p.i.'s constructed by the other distributions since different values of μ cause different estimates for α and β when a Gamma or a Weibull distribution is fitted because the value of their location parameter is forced to have value 0.

Normal, log-normal, and distribution-free prediction intervals perform very well in terms of the actual coverage even when they are constructed using small samples. The performance of normal p.i.'s is quite good even with sample sizes as small as 10. The variation of actual coverage caused by the difference in sample size is trivial and erratic, and thus seems to be a sampling variation.

The coverage performance of Weibull p.i.'s is quite poor over the whole range of confidence level with sample size 10. But their performance quickly improves as the sample size increases, showing a satisfactory performance when sample size is 30.

The average of the interval length of distribution-free p.i.'s cannot be said to be larger than that of normal p.i.'s (Table 4). But the variation is somewhat larger (Table 5).

2) Gamma Distributions

Experiments are done using sample size 30 for a wide range of the value of shape parameter. Several sample sizes are applied to an exponential distribution, a special case of the Gamma and the Weibull distributions in order to measure the sample size effect.

For large values of the shape parameter (e.g., $\alpha > 9.0$), all the types of p.i.'s except Gamma p.i.'s show a good coverage performance for the confidence level $< 95\%$. But only normal p.i.'s and log-normal p.i.'s perform well at the confidence level 99%. Surprisingly, Gamma p.i.'s perform worst among all the types of p.i.'s considered (Table 6).

Normal p.i.'s work well with large α, but show a singular deviation pattern as the value of α decreases. Patterns similar to those shown in Table 1 are found again with sample size 30. If normal p.i.'s result in an overly actual coverage for some combination of α value and confidence level, an increase in the sample size will aggravate the bad performance.

The actual coverage of Gamma p.i.'s is less than desired for the whole range of confidence level regardless of the value of α. Also, it is lower than those of the other types of p.i.'s as long as $\alpha > 1.0$. For $\alpha < 1.0$, only Weibull p.i.'s show a lower actual coverage than Gamma p.i.'s. In this context it can be said that the performance of Gamma p.i.'s is the poorest for a wide range of the shape of the population distribution when the size of the used sample is 30. But the coverage performance of Gamma p.i.'s is the most stable with respect to a change in the value of α. Also, if the sample size increases, the actual coverage of Gamma p.i.'s converges to the desired level, which is evidenced by Table 6.

Weibull p.i.'s perform satisfactorily for the confidence level not close to 100%

for a wide range of α values. The actual coverage curve of Weibull p.i.'s shows a downward shift as α decreases. But the speed of this degradation is μch slower than that shown by normal p.i.'s. The actual coverage curve of Weibull p.i.'s lie above that of Gamma p.i.'s as long as $\alpha < 1.0$ At $\alpha = 1.0$, both curve coincide very well. For $\alpha < 1.0$, Weibull p.i.'s are slightly inferior to Gamma p.i.'s.

In general the coverage performance of log-normal p.i.'s is quite satisfactory. It is no worse than those of the other types of p.i.'s until α decreases down to 1.0. But, if α continues to drop past 1.0, log-normal p.i.'s show a considerable coverage deviation in the same direction as that shown by normal p.i.'s, though the degree of malperformance is μch less serious.

As expected, distribution-free p.i.'s consistently show a good coverage performance regardless of the value of α and sample size as long as they allow the p.i. to fall in the range of the data.

Interval lengths of Gamma and Weibull p.i.'s are shorter than those of the other p.i.'s for the presented range of α and confidence level. Their good length performance becomes more distinct as the value of α approaches 0(Table 7,8). The only exception is the case with $\alpha = 0.5$ and $r = 99\%$ where normal p.i.'s show a shorter interval length with a very poor actual coverage. The average interval length of Gamma p.i.'s is shorter than that of Weibull p.i.'s when Gamma p.i.'s are inferior to Weibull p.i.'s in terms of actual coverage, and vice versa. The interval length of log-normal p.i.'s is extremely long and variable especially for small values of α and confidence levels close to 100%. The length performance of distribution-free p.i.'s is worse than those of Gamma p.i.'s and Weibull p.i.'s. But it is comparable to that of normal p.i.'s, or even better than that for low confidence level and low α value.

3) Weibull distributions

Experiments are executed for the sample size 30 and various values of the shape parameter α. The performance of Gamma p.i.'s is measured only for the Weibull distributions with $0.5 < \alpha < 3.0$ due to the calculational limitation on estimating the shape parameter value to fit a Gamma distribution.

If the same sample size is used, the actual coverage of Weibull and log-normal p.i.'s is independent of the value of the shape parameter α (Table 9). Also, the actual coverage of distribution-free p.i.'s is very stable with respect to a change in the value of α. Gamma p.i.'s and Weibull p.i.'s show a very similar coverage pattern in the whole range of confidence level for α = 1, 2, 3.

It is only normal p.i.'s that show a serious coverage gap for a wide range of confidence level for $\alpha < 1.0$. But the coverage performance of normal p.i.'s is comparable to those of the other p.i.'s for $\alpha > 2$. The coverage gap shown for small values of α has a pattern similar to that found in case of Gamma population distributions.

The actual coverage curve of Weibull p.i.'s is located below those of the other p.i.'s for all values of α when the size of used samples is 30. This malperformance µst have been caused by the fact that the approximation forµlas for Weibull p.i.'s are vulnerable to a small sample size. Examination of the cases where n = 5, 10, 40 supports this assertion. As seen in Table 9, the undesirable effect of small sample size seems to have vanished when the sample size is 40. Together with Gamma p.i.'s, Weibull p.i.'s show the best length performance.

Log-normal p.i.'s perform considerably well in terms of the actual coverage, though they overestimate for the confidence level 99%. But they are pretty bad in terms of the average and variation of the interval length when the value of α is close to 0 and the confidence level is close to 100% (Tables are not presented). It doesn't seem that their performance get improved by using larger samples.

4) Log-normal Distributions

With sample size fixed, the actual coverage of all types of p.i.'s is independent of the value of μ. Therefore the analysis is focused on the effect of sample size on the performance of each type of p.i.'s using only the log-normal distribution with μ = 1 and ^{t} = 1.

Normal p.i.'s fail to predict well for this distribution regardless of the sample size (Table 10). Actually, increasing the sample size makes the problem more serious.

Log-normal p.i.'s work perfectly over the whole range of sample size and confidence level. This is true for distribution-free p.i.'s as long as the p.i. falls upon inside the range of the data. But distribution-free p.i.'s are inferior to log-normal p.i.'s in terms of the average and variation of the interval length (Tables are not presented).

It can be said that Gamma and Weibull p.i.'s performs similarly, though the actual coverage of Gamma p.i.'s is lower than that of Weibull p.i.'s. Their actual coverage is lower than designed with small samples, but increases in accordance with the increase of the sample size. Average and variation of the length of Gamma p.i.,'s are smaller than those of Weibull p.i.'s.

5) Delay in an MMI Queue

Recall that the analysis is on the mean delay time of the first 25 customers in an MMl queue with the traffic intensity 0.9 initiated empty and idle.

With small samples, all the types of p.i.'s result in an actual coverage lower than the designed level for the whole range of confidence level. Even the distribution-free p.i.'s work poorly (Table 11). But their performance gets improved as the sample size increases.

In case of normal p.i.'s, the speed of increase in the actual coverage caused by increasing sample size is different for different confidence level so that a serious overestimation remains in the range of confidence level greater than 90% while a considerable underestimation takes place in a range including 70% and 80%.

Gamma p.i.'s and Weibull p.i.'s perform in a similar pattern.

The actual coverage of log-normal p.i.'s is better than those of Gamma and Weibull p.i.'s when small samples (n = 5, 10) are used. But this relative advantage becomes less obvious as the sample size increases so that it is difficult to mention their relative strength when samples of size 100 are used. The problem with log-normal p.i.'s is their interval length. The average and variation in the length is obviously beyond the scope of acceptance.

6) Reliability Model

Just one example of reliability model is analyzed with varying sample size : 5, 10, 20, 40, 100.

Graph 3 makes distinct the comparative strengths of each p.i. construction procedure by showing a case where considerably large samples are used. It shows that Gamma, Weibull, and distribution-free p.i.'s work satisfactorily if the desired coverage is not pretty close to 100%. Log-normal p.i.'s are not too bad even though they underestimate the coverage for a wide range of confidence level including both 70% and 90%.

The actual coverage curve of Gamma p.i.'s and Weibull p.i.'s shift downward as the used sample size decreases so that they overestimate the coverage for all the range of confidence level if the sample size is as small as 20. It can be said that Gamma p.i.'s are slightly better than Weibull p.i.'s for the whole range of confidence level except in the narrow range close to 100% where Weibull p.i.'s yield a better actual coverage. But, in general, both of them show very similar coverage pattern with all the considered sample sizes (Table 12).

The actual coverage of normal p.i.'s decreases as the sample size decreases. So, the coverage gap happens to decrease for the range of confidence level where a serious underestimation existed with large samples. But we should not count on this phenomenon because the decrease in the actual coverage aggravates the overestimation of the coverage for the range of confidence level likely to be of interest. In a word, normal p.i.'s are not reliable.

The coverage performance of log-normal p.i.'s looks μch better with smaller samples than with larger samples(Compare the case of $n = 5$ with that of $n = 100$. This is so because the actual coverage is higher than designed with large samples and a decrease in the sample size doesn't lower the actual coverage curve so μch. We want to treat this fact just as an incident rather than treating small samples as a remedy for underestimation. Once again, the average and variability in the interval length turn out to be the serious defect of log-normal p.i.'s. They are several times larger than those of the other types of p.i.'s even with large samples for confidence level 95% and becomes absolutely intolerable as the sample size decreases(The relevant tables are omitted).

V. Conclusions

We can summarize the observations described so far as follows Normal p.i.'s lack in the capability to fit to a variety of population distributions. They work well only when the population distribution itself has a normal distribution or a distribution close to normal. If the population has a skewed distribution, they yield inappropriately wide intervals underestimating the coverage for low confidence levels, but overly narrow intervals with degenerated actual coverage for high confidence levels.

Gamma p.i.'s and Weibull p.i.'s show similar performance. Gamma p.i.'s work better for some population distribution types, Weibull p.i.'s do for others. Their performance in case of normal distributions is quite satisfactory if large enough samples are used. Also, they yield a good performance for diverse population distributions. The caveat is to use large samples because the construction procedures of these p.i.'s are based on approximation forμlas which are vulnerable to small samples. But sample size of 30 - 40 would be large enough to avoid the negative effect of small samples for most of the population distributions.

Log-normal p.i.'s also show a good coverage performance for diverse population distributions. But it is obtained at the expense of huge and volatile interval length in most of the cases.

Distribution-free p.i.'s are steadily reliable in terms of actual coverage if the sample size and confidence level allow them to nest in the range of the data. The only possible source of concern is their interval length, which is traditionally believed to be longer and μch more variable than those of parametric p.i.'s. But this concern shouldn't be blindly emphasized. In many cases their interval length is comparable to or even better than those of the other types of p.i.'s in terms of average and variance. Especially, their interval length shortens and quickly gets stable with an increase of the number of data thrown out over the boundaries. For example, if we want to construct a p.i. with confidence level 90% using a sample of 40 observations, there's no reason to avoid using a distribution-free p.i. due to a doubt about its length performance. Each boundary has two data points outside.

The coverage gap doesn't vanish by using larger samples if a wrong distribution is fitted to the data even in the cases where the gap is not serious with small samples. Generally, a procedure shows the best length performance when it is based on the right fitted distribution. Unless the exact type of the population distribution is known, there's no hope to clearly reveal the characteristics of the tails with small samples. We must rely on luck if we construct a p.i. with confidence level 99% using, for example, a sample of 30 observations and expect the coverage gap is less than 1%.

The above summary leads us to conclude our analysis by recommending the following procedure. Don't forget it is assumed hereafter throughout the end that we don't have abundant data. Also, it is assumed that we have no a priori knowledge on the population distribution. Suppose we want to make an interval estimate for the next observation using a given sample.

1) Check if the sample size and desired coverage allow us a p.i. nested in the range of the data. If so, resort to the distribution-free interval estimation procedure. Otherwise, go to the next step.

2) Check if the sample size is around or larger than 30. If so, rely on Gamma p.i.'s or Weibull p.i.'s unless the data strongly advocate for fitting a normal distribution in which case fitting a normal distribution is allowed. Otherwise, go to the next step.

3) There's no alternative but to rely on the p.i. construction procedure based on the normality assumption which may sometimes produce a crazy interval which conservatively estimates the coverage as long as the designed coverage is modest(e.g. less than 90%).

We want to add a few warnings.

Forget about prediction intervals with inappropriately high confidence level. For example, it is quite risky to form a p.i. with confidence level 99% using a sample of size 30.

Don't forge the data to make it to be better fitted with a normal distribution. For example, don't throw out outlying data points which may be a signal to expose a fat tail. Instead fit a distribution more flexible to such data.

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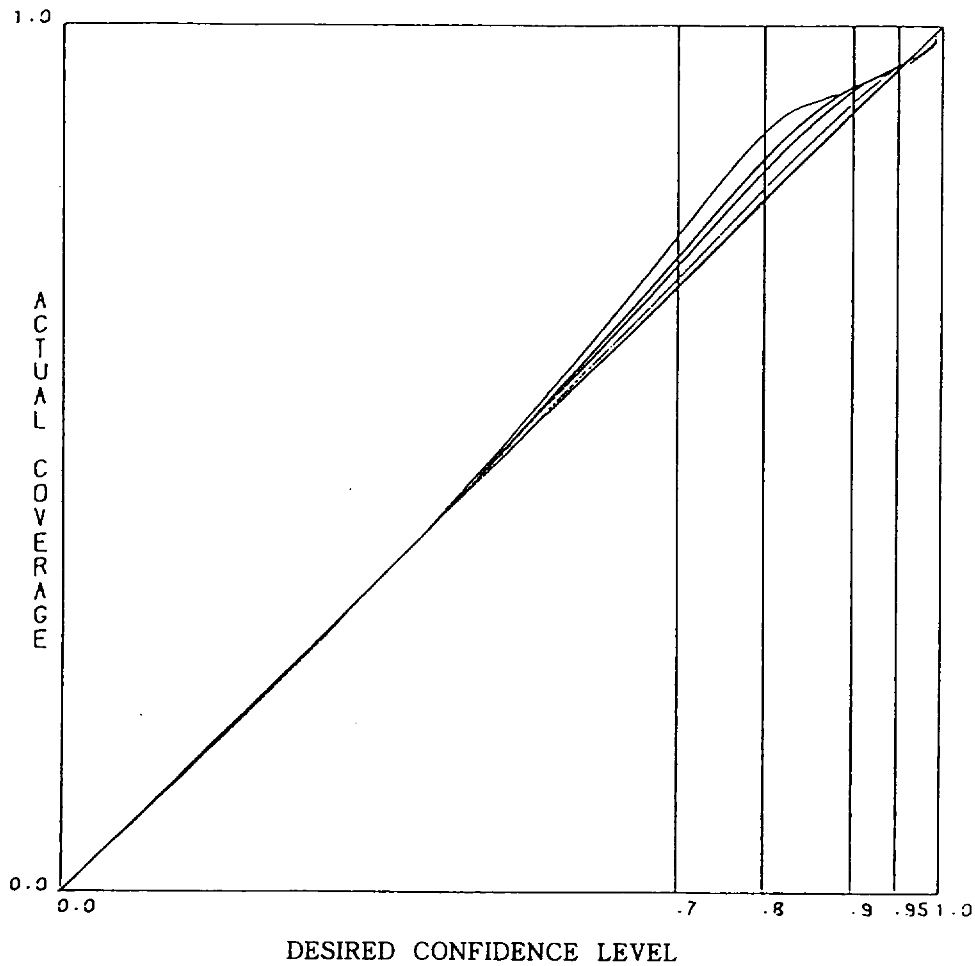
APPENDIX

Graph 1 : Actual Coverage of Normal P. I. 's



POPULATION DISTRIBUTION : GAMMA

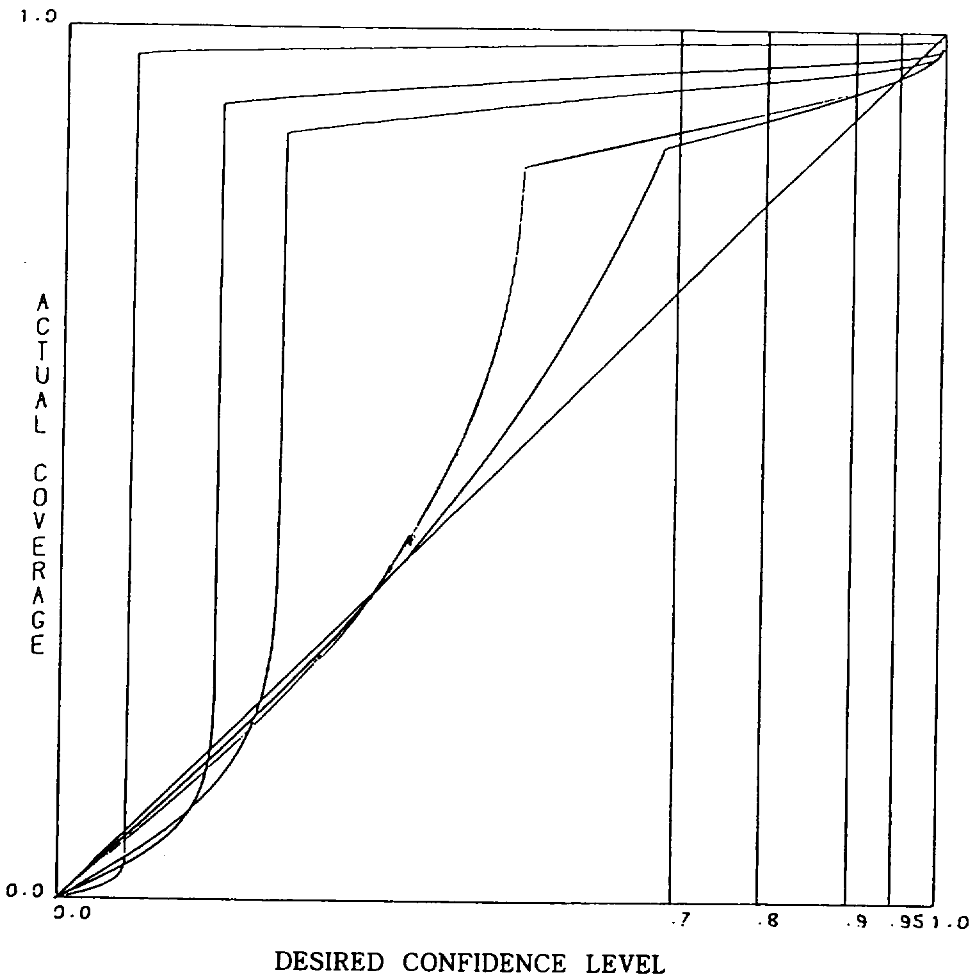
DISTRIBUTION PARAMETERS : 50.0, 9.0, 4.0, 3.0, 2.0

FITTED DISTRIBUTION : NORMAL

SAMPLE SIZE : INFINITE

NO. OFCONFIDENCE LEVELS : 200

Graph 2 : Actual Coverage of Normal P. I. 's



POPULATION DISTRIBUTION : GAMMA

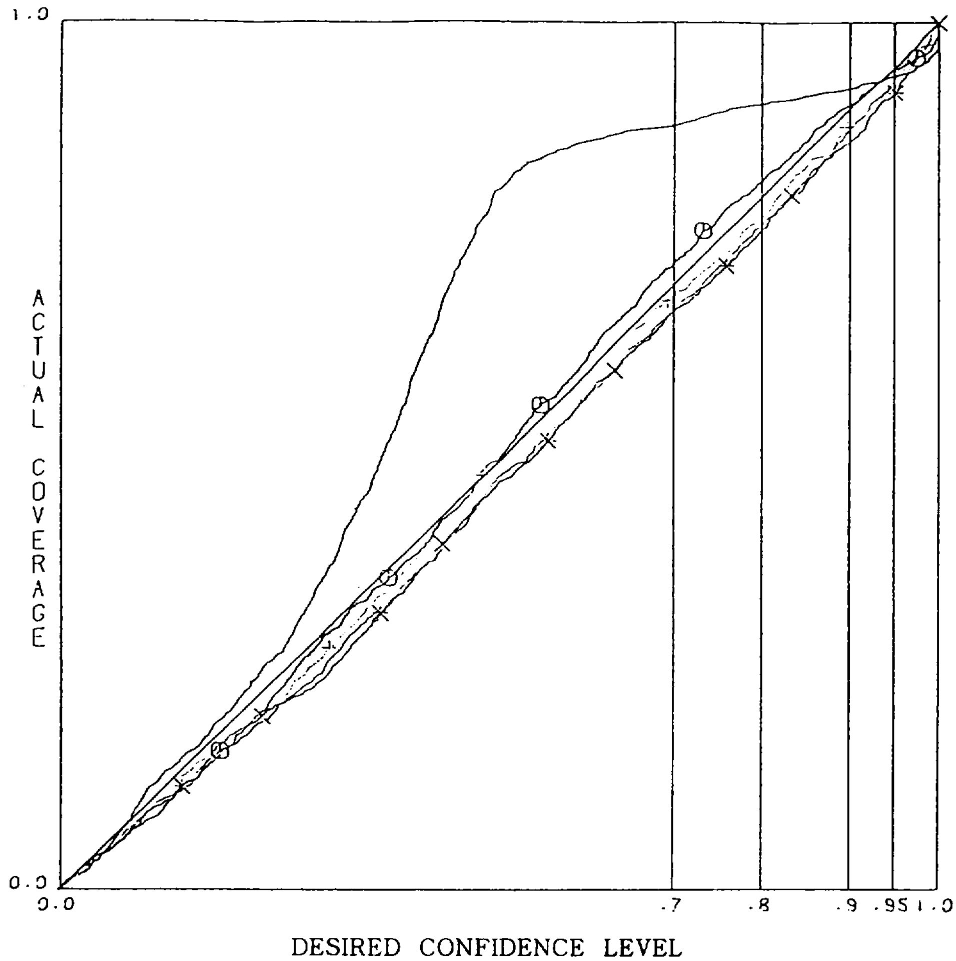
DISTRIBUTION PARAMETERS : 1.0, 0.5, 0.1, 0.05, 0.01

FITTED DISTRIBUTION : NORMAL

SAMPLE SIZE : INFINITE

NO. OFCONFIDENCE LEVELS : 500

Graph 3 : Observed Coverage of P. I. Construction Procedures



POPULATION DISTRIBUTION : RELIABILITY MODEL

DISTRIBUTION PARAMETERS : 0.5, 1.0

SAMPLE SIZE : 100, REPLICATIONS : 1000

FITTED DISTRIBUTION : NORMAL

GAMMA

WEIBULL

LOGNORMAL

DIST. FREE

Table 1 : Observed Coverage of the Normal P.I.'s

Population Distribution : Gamma(α, 1.0, 0.0)

No. of Replications : 1,000

ALPHA VALUE	GOAL COVERAGE	SAMPLE SIZE						
		5	10	20	30	40	100	INFINTE
50.00	0.7000	0.7026	0.7003	0.7026	0.7034	0.7031	0.7015	0.7018
	0.8000	0.8011	0.8006	0.8030	0.8040	0.8036	0.8022	0.8024
	0.9000	0.8987	0.8999	0.9022	0.9032	0.9031	0.9019	0.9023
	0.9500	0.9480	0.9491	0.9508	0.9517	0.9517	0.9509	0.9512
	0.9900	0.9889	0.9890	0.9892	0.9895	0.9895	0.9891	0.9892
9.00	0.7000	0.6982	0.7048	0.7046	0.7066	0.7075	0.7096	0.7103
	0.8000	0.7968	0.8063	0.8073	0.8093	0.8106	0.8133	0.8142
	0.9000	0.8926	0.9023	0.9047	0.9067	0.9081	0.9114	0.9125
	0.9500	0.9414	0.9475	0.9492	0.9506	0.9517	0.9543	0.9553
	0.9900	0.9853	0.9853	0.9843	0.9844	0.9847	0.9853	0.9853
4.00	0.7000	0.7134	0.7153	0.7172	0.7188	0.7204	0.7228	0.7252
	0.8000	0.8098	0.8172	0.8225	0.8253	0.8277	0.8313	0.8344
	0.9000	0.8974	0.9056	0.9121	0.9155	0.9182	0.9224	0.9261
	0.9500	0.9418	0.9442	0.9481	0.9502	0.9520	0.9538	0.9553
	0.9900	0.9839	0.9794	0.9794	0.9797	0.9802	0.9805	0.9809
2.00	0.7000	0.7232	0.7372	0.7469	0.7486	0.7508	0.7550	0.7597
	0.8000	0.8156	0.8354	0.8496	0.8547	0.8588	0.8681	0.8781
	0.9000	0.8941	0.9071	0.9167	0.9200	0.9225	0.9267	0.9296
	0.9500	0.9340	0.9389	0.9437	0.9450	0.9465	0.9489	0.9511
	0.9900	0.9777	0.9733	0.9732	0.9735	0.9741	0.9752	0.9765
1.00	0.7000	0.7484	0.7816	0.8028	0.8142	0.8224	0.8427	0.8695
	0.8000	0.8246	0.8556	0.8763	0.8823	0.8866	0.8942	0.8979
	0.9000	0.8859	0.9053	0.9161	0.9187	0.9209	0.9258	0.9290
	0.9500	0.9188	0.9310	0.9377	0.9396	0.9414	0.9454	0.9482
	0.9900	0.9622	0.9637	0.9660	0.9667	0.9677	0.9702	0.9720
0.50	0.7000	0.7788	0.8230	0.8568	0.8647	0.8711	0.8781	0.8836
	0.8000	0.8334	0.8656	0.8859	0.8921	0.8951	0.9011	0.9065
	0.9000	0.8760	0.8986	0.9133	0.9187	0.9215	0.9269	0.9318
	0.9500	0.9034	0.9191	0.9317	0.9363	0.9387	0.9435	0.9479
	0.9900	0.9452	0.9493	0.9573	0.9604	0.9621	0.9655	0.9688
0.10	0.7000	0.8246	0.8697	0.8995	0.9097	0.9149	0.9245	0.9318
	0.8000	0.8372	0.8804	0.9100	0.9200	0.9252	0.9348	0.9420
	0.9000	0.8522	0.8939	0.9225	0.9323	0.9373	0.9466	0.9536
	0.9500	0.8644	0.9038	0.9314	0.9408	0.9457	0.9546	0.9613
	0.9900	0.8870	0.9206	0.9453	0.9538	0.9583	0.9663	0.9723
0.01	0.7000	0.7736	0.8962	0.9405	0.9543	0.9610	0.9738	0.9830
	0.8000	0.7750	0.8978	0.9421	0.9560	0.9627	0.9755	0.9847
	0.9000	0.7770	0.8998	0.9442	0.9580	0.9647	0.9775	0.9867
	0.9500	0.7787	0.9013	0.9456	0.9595	0.9661	0.9789	0.9880
	0.9900	0.7819	0.9040	0.9481	0.9618	0.9685	0.9812	0.9901

Table 2 : Observed Coverage of the Normal P. I.'s

Population Distribution : Weibull(α, 1.0, 0.0)

No. of Replications : 1,000

ALPHA VALUE	GOAL COVERAGE	SAMPLE SIZE						
		5	10	20	30	40	100	INFINTE
50.00	0.7000	0.7066	0.7124	0.7191	0.7253	0.7274	0.7305	0.7348
	0.8000	0.8023	0.8119	0.8190	0.8254	0.8277	0.8312	0.8358
	0.9000	0.8932	0.9018	0.9074	0.9125	0.9145	0.9176	0.9215
	0.9500	0.9399	0.9431	0.9458	0.9491	0.9502	0.9520	0.9544
	0.9900	0.9832	0.9798	0.9787	0.9795	0.9797	0.9798	0.9805
9.00	0.7000	0.6998	0.7004	0.7022	0.7058	0.7068	0.7074	0.7095
	0.8000	0.7988	0.8023	0.8043	0.8079	0.8089	0.8096	0.8118
	0.9000	0.8959	0.9005	0.9022	0.9054	0.9063	0.9071	0.9090
	0.9500	0.9453	0.9475	0.9482	0.9503	0.9509	0.9515	0.9529
	0.9900	0.9875	0.9862	0.9852	0.9856	0.9857	0.9855	0.9858
4.00	0.7000	0.6946	0.6908	0.6882	0.6895	0.6897	0.6887	0.6894
	0.8000	0.7967	0.7948	0.7921	0.7932	0.7934	0.7923	0.7930
	0.9000	0.8989	0.9002	0.8982	0.8991	0.8992	0.8983	0.8988
	0.9500	0.9501	0.9525	0.9516	0.9523	0.9524	0.9519	0.9523
	0.9900	0.9907	0.9919	0.9922	0.9926	0.9928	0.9930	0.9934
3.00	0.7000	0.6948	0.6901	0.6860	0.6864	0.6865	0.6851	0.6856
	0.8000	0.7978	0.7955	0.7916	0.7919	0.7919	0.7905	0.7909
	0.9000	0.9000	0.9020	0.9000	0.9004	0.9005	0.8996	0.9000
	0.9500	0.9503	0.9539	0.9539	0.9545	0.9547	0.9547	0.9552
	0.9900	0.9906	0.9921	0.9927	0.9930	0.9932	0.9937	0.9941
2.00	0.7000	0.7021	0.6994	0.6944	0.6940	0.6940	0.6927	0.6934
	0.8000	0.8044	0.8077	0.8056	0.8056	0.8059	0.8052	0.8061
	0.9000	0.8994	0.9075	0.9109	0.9128	0.9142	0.9164	0.9186
	0.9500	0.9458	0.9514	0.9544	0.9555	0.9565	0.9587	0.9600
	0.9900	0.9869	0.9871	0.9869	0.9865	0.9865	0.9867	0.9868
1.00	0.7000	0.7484	0.7816	0.8028	0.8142	0.8224	0.8427	0.8695
	0.8000	0.8246	0.8556	0.8763	0.8823	0.8866	0.8942	0.8979
	0.9000	0.8859	0.9053	0.9161	0.9187	0.9209	0.9258	0.9290
	0.9500	0.9188	0.9310	0.9377	0.9396	0.9414	0.9454	0.9482
	0.9900	0.9622	0.9637	0.9660	0.9667	0.9677	0.9702	0.9720
0.50	0.7000	0.7995	0.8532	0.8842	0.8926	0.8982	0.9116	0.9239
	0.8000	0.8324	0.8766	0.9005	0.9083	0.9136	0.9263	0.9380
	0.9000	0.8619	0.8988	0.9192	0.9262	0.9310	0.9424	0.9531
	0.9500	0.8827	0.9138	0.9319	0.9382	0.9424	0.9527	0.9624
	0.9900	0.9186	0.9373	0.9506	0.9553	0.9586	0.9668	0.9747
0.10	0.7000	0.8249	0.8972	0.9371	0.9510	0.9594	0.9781	0.9998
	0.8000	0.8295	0.9007	0.9398	0.9534	0.9615	0.9795	0.9998
	0.9000	0.8365	0.9051	0.9431	0.9562	0.9640	0.9811	0.9998
	0.9500	0.8416	0.9085	0.9454	0.9582	0.9657	0.9822	0.9999
	0.9900	0.8518	0.9142	0.9492	0.9613	0.9684	0.9839	0.9999

Table 3 : Observed Coverage of P. I. Construction Procedures (1)

Population Distribution : Normal(μ, 1.0)

No. of Replications : 1,000

μ	Sample Size	Desired coverage (Confidence level)				
		0.70000	0.80000	0.90000	0.95000	0.99000
5.0	10	0.70921	0.80832	0.90602	0.95397	0.99132
		0.00000	0.00000	0.00000	0.00000	0.00000
		0.64839	0.74412	0.84428	0.89929	0.95462
		0.72087	0.81728	0.90768	0.95043	0.98587
		0.72905	0.81649	0.00000	0.00000	0.00000
5.0	30	0.69947	0.79947	0.89958	0.94973	0.98996
		0.00000	0.00000	0.00000	0.00000	0.00000
		0.69599	0.79281	0.88836	0.93628	0.97815
		0.71647	0.81494	0.90752	0.95019	0.98404
		0.69999	0.79934	0.91027	0.00000	0.00000
5.0	40	0.69903	0.79894	0.89900	0.94919	0.98966
		0.00000	0.00000	0.00000	0.00000	0.00000
		0.70300	0.79971	0.89414	0.94074	0.98049
		0.71656	0.81495	0.90739	0.94994	0.98370
		0.69859	0.79990	0.90190	0.94939	0.00000
5.0	100	0.69935	0.79937	0.89949	0.94963	0.98987
		0.00000	0.00000	0.00000	0.00000	0.00000
		0.71689	0.81347	0.90580	0.94985	0.98547
		0.71943	0.81799	0.91016	0.95192	0.98403
		0.69911	0.79935	0.90077	0.95221	0.00000
4.0	30	0.69947	0.79947	0.89958	0.94973	0.98996
		0.00000	0.00000	0.00000	0.00000	0.00000
		0.69059	0.78714	0.88375	0.93338	0.97799
		0.72826	0.82479	0.91176	0.95015	0.98106
		0.69999	0.79934	0.91027	0.00000	0.00000

Table 4 : Average of the P. I. Length of P. I. Construction Procedures

Population Distribution : Normal(μ, 1.0)

No. of Replications : 1,000

μ	Sample Size	Desired coverage (Confidence level)				
		0.70000	0.80000	0.90000	0.95000	0.99000
5.0	10	2.28423	2.87270	3.80757	4.69875	6.75026
		0.00000	0.00000	0.00000	0.00000	0.00000
		2.00047	2.47015	3.15891	3.74457	4.83664
		2.36303	2.99213	4.02177	5.04782	7.63936
		2.42286	3.02669	0.00000	0.00000	0.00000
5.0	30	2.12573	2.64167	3.42261	4.11978	5.55229
		0.00000	0.00000	0.00000	0.00000	0.00000
		2.11165	2.60653	3.33106	3.94564	5.08649
		2.21050	2.76047	3.61027	4.39105	6.08108
		2.15168	2.68506	3.62824	0.00000	0.00000
5.0	40	2.11042	2.61922	3.38518	4.06391	5.44062
		0.00000	0.00000	0.00000	0.00000	0.00000
		2.12933	2.62820	3.35841	3.97760	5.12624
		2.19494	2.73693	3.56911	4.32694	5.94071
		2.12705	2.66054	3.50309	4.27602	0.00000
5.0	100	2.08583	2.58286	3.32406	3.97234	5.25799
		0.00000	0.00000	0.00000	0.00000	0.00000
		2.16413	2.67083	3.41209	4.04013	5.20347
		2.17840	2.70944	3.51607	4.23960	5.73847
		2.09236	2.59637	3.36590	4.09194	0.00000
4.0	30	2.12573	2.64167	3.42261	4.11977	5.55229
		0.00000	0.00000	0.00000	0.00000	0.00000
		2.08659	2.56867	3.26677	3.85052	4.91065
		2.27715	2.85388	3.75805	4.60539	6.50421
		2.15168	2.68506	3.62824	0.00000	0.00000

Table 5 : Standard Deviation of the P. I. Length of P. I. Construction Procedures

Population Distribution : Normal(μ, 1.0)

No. of Replications : 1,000

μ	Sample Size	Desired coverage (Confidence level)				
		0.70000	0.80000	0.90000	0.95000	0.99000
5.0	10	0.53744	0.67589	0.89586	1.10552	1.58823
		0.00000	0.00000	0.00000	0.00000	0.00000
		0.47621	0.58121	0.72739	0.84297	1.03284
		0.61835	0.79398	1.09772	1.42443	2.37565
		0.59555	0.73788	0.00000	0.00000	0.00000
5.0	30	0.28204	0.35052	0.45410	0.54664	0.73665
		0.00000	0.00000	0.00000	0.00000	0.00000
		0.28689	0.34973	0.43673	0.50498	0.61515
		0.33599	0.42383	0.56501	0.70155	1.02372
		0.35716	0.43375	0.53204	0.00000	0.00000
5.0	40	0.25028	0.31062	0.40149	0.48196	0.64525
		0.00000	0.00000	0.00000	0.00000	0.00000
		0.26103	0.31822	0.39738	0.45946	0.55982
		0.29391	0.36996	0.49104	0.60673	0.87332
		0.31924	0.38940	0.49245	0.64962	0.00000
5.0	100	0.15178	0.18795	0.24188	0.28913	0.38263
		0.00000	0.00000	0.00000	0.00000	0.00000
		0.16242	0.19789	0.24693	0.28536	0.34709
		0.19142	0.24028	0.31729	0.38966	0.55130
		0.19515	0.23407	0.28749	0.37540	0.00000
4.0	30	0.28204	0.35050	0.45411	0.54665	0.73666
		0.00000	0.00000	0.00000	0.00000	0.00000
		0.27783	0.33767	0.41959	0.48303	0.58477
		0.39956	0.51116	0.69965	0.89377	1.39978
		0.35716	0.43475	0.53204	0.00000	0.00000

Table 6 : Observed Coverage of P. I. Construction Procedures (2)

Population Distribution : Normal(α, 1.0, 0.0)

No. of Replications : 1,000

α	Sample Size	Desired coverage (Confidence level)				
		0.70000	0.80000	0.90000	0.95000	0.99000
16.0	30	0.70544	0.80737	0.90615	0.95242	0.98758
		0.00000	0.00000	0.00000	0.00000	0.00000
		0.71551	0.81248	0.90243	0.94331	0.97658
		0.70327	0.80327	0.90223	0.95089	0.98934
		0.70382	0.80154	0.90916	0.00000	0.00000
9.0	30	0.70898	0.81196	0.90900	0.95227	0.98519
		0.67539	0.77467	0.87725	0.93174	0.98116
		0.70985	0.80703	0.89895	0.94192	0.97723
		0.70348	0.80339	0.90179	0.94988	0.98815
		0.70235	0.80272	0.90875	0.00000	0.00000
4.0	30	0.71907	0.82540	0.91536	0.95010	0.97962
		0.67285	0.77217	0.87513	0.93010	0.98037
		0.69668	0.79458	0.89054	0.93774	0.97780
		0.70609	0.80606	0.90277	0.94885	0.98562
		0.69869	0.79823	0.90569	0.00000	0.00000
1.0	30	0.81585	0.88492	0.92093	0.94145	0.96792
		0.67431	0.77349	0.87611	0.93075	0.98060
		0.67368	0.77344	0.87665	0.93139	0.98076
		0.73151	0.83016	0.91386	0.94840	0.97796
		0.69851	0.80178	0.90616	0.00000	0.00000
0.5	30	0.86758	0.89242	0.91919	0.93678	0.96090
		0.67413	0.77337	0.87607	0.93053	0.97812
		0.65997	0.76329	0.87168	0.92925	0.98067
		0.75461	0.85009	0.91929	0.94601	0.97317
		0.69827	0.79910	0.90091	0.00000	0.00000
1.0	5	0.75297	0.82909	0.89194	0.92606	0.96784
		0.00000	0.00000	0.00000	0.00000	0.00000
		0.55393	0.64565	0.75091	0.81691	0.89747
		0.71529	0.80956	0.89652	0.94016	0.98258
		0.00000	0.00000	0.00000	0.00000	0.00000
1.0	10	0.77348	0.85495	0.90604	0.93041	0.96283
		0.62135	0.71773	0.82323	0.88503	0.95241
		0.61908	0.71668	0.82364	0.88587	0.95218
		0.71554	0.81313	0.90060	0.94082	0.97788
		0.70824	0.79812	0.00000	0.00000	0.00000
1.0	100	0.84272	0.89421	0.92578	0.94542	0.97017
		0.69123	0.79110	0.89223	0.94396	0.98735
		0.69100	0.79104	0.89232	0.94405	0.98730
		0.73697	0.83691	0.91991	0.95196	0.97871
		0.69861	0.79882	0.89858	0.94989	0.00000

Table 7 : Average of the P. I. Length of P. I. Construction Procedures

Population Distribution : Gamma (α, 1.0, 0.0)

No. of Replications : 1,000

α	Sample Size	Desired coverage (Confidence level)				
		0.70000	0.80000	0.90000	0.95000	0.99000
16.0	30	8.50052	10.56368	13.68657	16.47444	22.20286
		0.00002	0.00000	0.00000	0.00000	0.00000
		8.67242	10.67113	13.55998	15.96989	20.33115
		8.49460	10.62949	13.95559	17.04606	23.86838
		8.57921	10.69083	14.37724	0.00000	0.00000
9.0	30	6.36339	7.90785	10.24560	12.33258	16.62078
		5.94862	7.36897	9.49045	11.35001	15.05071
		6.36762	7.81031	9.87014	11.56328	14.57153
		6.33551	7.97230	10.57931	13.07417	18.86562
		6.36881	7.99318	10.73075	0.00000	0.00000
4.0	30	4.21263	5.23507	6.78269	8.16429	11.00314
		3.85995	4.79372	6.20290	7.45531	10.00576
		4.03747	4.94385	6.23414	7.29637	9.21829
		4.18115	5.35276	7.33777	9.39229	14.80178
		4.12134	5.17869	7.07330	0.00000	0.00000
1.0	30	2.08145	2.58664	3.35131	4.03395	5.43661
		1.69782	2.15014	2.87978	3.58068	5.16450
		1.69468	2.14508	2.87117	3.56886	5.14942
		2.06213	2.98939	5.12578	8.25182	22.23766
		1.82634	2.35104	3.39537	0.00000	0.00000
0.5	30	1.43061	1.77783	2.30340	2.77259	3.73667
		0.99684	1.31427	1.86984	2.44186	3.81725
		0.97108	1.33318	2.02975	2.82579	5.04309
		1.54912	2.90562	7.96866	21.52253	254.06108
		1.09769	1.48277	2.30625	0.00000	0.00000
1.0	5	2.25391	2.90501	4.03927	5.26061	8.72351
		0.00000	0.00000	0.00000	0.00000	0.00000
		1.46953	1.86130	2.49518	3.10823	4.51900
		2.79418	5.01791	15.97499	77.32551	*********
		0.00000	0.00000	0.00000	0.00000	0.00000
1.0	10	2.10535	2.64773	3.50940	4.33078	6.22163
		1.61132	2.03895	2.72649	3.38415	4.86114
		1.60334	2.02607	2.70443	3.35356	4.82046
		2.27517	3.48670	6.85293	13.41209	75.15778
		2.08476	2.68567	0.00000	0.00000	0.00000
1.0	100	2.06770	2.56041	3.29517	3.93783	5.21229
		1.72023	2.17861	2.91839	3.62969	5.23963
		1.71909	2.17648	2.91422	3.62331	5.22865
		1.98966	2.83559	4.68182	7.18361	16.69032
		1.76088	2.24057	3.02879	3.86167	0.00000

Table 8 : Standard Deviation of the P. I. Length of P. I. Construction Procedures

Population Distribution : Gamma(α, 1.0, 0.0)

No. of Replications : 1,000

α	Sample Size	Desired coverage (Confidence level)				
		0.70000	0.80000	0.90000	0.95000	0.99000
16.0	30	1.15816	1.43925	1.86475	2.24463	3.02499
		0.00000	0.00000	0.00000	0.00000	0.00000
		1.25976	1.53219	1.90672	2.19864	2.67392
		1.35658	1.43997	1.93500	2.42347	3.61248
		1.36872	1.67620	2.18393	0.00000	0.00000
9.0	30	0.95503	1.18682	1.53773	1.85088	2.49454
		0.83763	1.04089	1.34842	1.62264	2.18390
		0.94868	1.15340	1.43604	1.65969	2.04369
		0.92666	1.19072	1.64244	2.11473	3.36913
		1.08307	1.35678	1.80913	0.00000	0.00000
4.0	30	0.73499	0.91338	1.18338	1.42442	1.91967
		0.58638	0.73272	0.95901	1.16658	1.61111
		0.63727	0.78197	0.99201	1.17173	1.52742
		0.68594	0.91716	1.36086	1.88903	3.57776
		0.75672	0.95104	1.34413	0.00000	0.00000
1.0	30	0.51044	0.63434	0.82186	0.98927	1.33326
		0.32987	0.42374	0.58257	0.74349	1.13078
		0.32433	0.42226	0.59834	0.79033	1.29955
		0.52845	0.88332	1.93147	3.87688	16.00802
		0.42582	0.56436	0.88973	0.00000	0.00000
0.5	30	0.43337	0.53855	0.69775	0.83988	1.13191
		0.25867	0.34397	0.49953	0.66616	1.08228
		0.25489	0.35659	0.58063	0.87900	1.88860
		0.94526	2.84847	17.23549	93.51546	*********
		0.34602	0.47566	0.73642	0.00000	0.00000
1.0	5	1.25854	1.62211	2.25546	2.93744	4.87106
		0.00000	0.00000	0.00000	0.00000	0.00000
		0.71498	0.93632	1.34515	1.80855	3.12698
		2.14657	6.39012	55.25169	649.01941	*********
		0.00000	0.00000	0.00000	0.00000	0.00000
1.0	10	0.84918	1.06794	1.41548	1.74678	2.50946
		0.55733	0.71587	0.98356	1.25381	1.90101
		0.55260	0.71884	1.01726	1.34299	2.21472
		1.06457	2.02623	5.93551	17.27785	232.26535
		0.77946	1.11108	0.00000	0.00000	0.00000
1.0	10	0.29409	0.36418	0.46866	0.56004	0.74132
		0.18260	0.23485	0.32354	0.41367	0.63119
		0.18039	0.23574	0.33583	0.44519	0.73432
		0.27274	0.43980	0.89038	1.63220	5.22515
		0.22863	0.30242	0.45595	0.65219	0.00000

Table 9 : Observed Coverage of P. I. Construction Procedures (3)

Population Distribution : Weibull(α, 1.0, 0.0)

No. of Replications : 1,000

α	Sample Size	Desired coverage (Confidence level)				
		0.70000	0.80000	0.90000	0.95000	0.99000
9.0	30	0.70666	0.80833	0.90525	0.95005	0.98541
		0.00000	0.00000	0.00000	0.00000	0.00000
		0.67368	0.77344	0.87665	0.93139	0.98076
		0.73151	0.83016	0.91386	0.94840	0.97796
		0.69945	0.80242	0.90938	0.00000	0.00000
4.0	30	0.69014	0.79375	0.89931	0.95233	0.99263
		0.00000	0.00000	0.00000	0.00000	0.00000
		0.67368	0.77344	0.87665	0.93139	0.98076
		0.73151	0.83016	0.91386	0.94840	0.97796
		0.69929	0.80234	0.90886	0.00000	0.00000
3.0	30	0.68702	0.79245	0.90086	0.95483	0.99311
		0.68313	0.78427	0.88460	0.93415	0.97691
		0.67368	0.77344	0.87665	0.93139	0.98076
		0.73151	0.83016	0.91386	0.94840	0.97796
		0.69929	0.80234	0.90886	0.00000	0.00000
0.5	30	0.89526	0.91089	0.92848	0.94017	0.95688
		0.70306	0.79707	0.88755	0.92998	0.96466
		0.67368	0.77344	0.87665	0.93139	0.98076
		0.73151	0.83016	0.91386	0.94840	0.97796
		0.69779	0.80054	0.90399	0.00000	0.00000
0.2	30	0.93674	0.94203	0.94819	0.95245	0.95897
		0.00000	0.00000	0.00000	0.00000	0.00000
		0.67368	0.77344	0.87665	0.93139	0.98076
		0.73151	0.83016	0.91386	0.94840	0.97796
		0.69725	0.79535	0.90231	0.00000	0.00000
3.0	5	0.70085	0.80418	0.90583	0.95480	0.99235
		0.00000	0.00000	0.00000	0.00000	0.00000
		0.55393	0.64565	0.75091	0.81691	0.89747
		0.71529	0.80856	0.89652	0.94016	0.98258
		0.00000	0.00000	0.00000	0.00000	0.00000
3.0	10	0.68752	0.79315	0.90068	0.95356	0.99247
		0.00000	0.00000	0.00000	0.00000	0.00000
		0.61908	0.71668	0.82364	0.88587	0.95218
		0.71554	0.81313	0.90060	0.94082	0.97788
		0.71254	0.80111	0.00000	0.00000	0.00000
3.0	40	0.68722	0.79259	0.90104	0.95519	0.99343
		0.69087	0.79229	0.89167	0.93962	0.97935
		0.68050	0.78036	0.88282	0.93640	0.98344
		0.73467	0.83364	0.91668	0.95014	0.97847
		0.69900	0.80062	0.90097	0.94991	0.00000

Table 10 : Observed Coverage of P.I. Construction Procedures (4)

Population Distribution : Log-normal(1.0, 1.0)

No. of Replications : 1,000

Sample Size	Desired coverage (Confidence level)				
	0.70000	0.80000	0.90000	0.95000	0.99000
5	0.76759	0.83172	0.88332	0.91146	0.95109
	0.00000	0.00000	0.00000	0.00000	0.00000
	0.57539	0.66821	0.76946	0.82845	0.89438
	0.70237	0.80230	0.90268	0.95309	0.99241
	0.00000	0.00000	0.00000	0.00000	0.00000
10	0.81903	0.87016	0.90511	0.92488	0.95205
	0.65604	0.75295	0.84968	0.89998	0.95008
	0.67305	0.76891	0.86141	0.90675	0.94884
	0.70921	0.80832	0.90602	0.95397	0.99132
	0.73025	0.81555	0.00000	0.00000	0.00000
20	0.85553	0.89443	0.92065	0.93630	0.95793
	0.69445	0.79292	0.88571	0.92944	0.96787
	0.71965	0.81460	0.89924	0.93637	0.96793
	0.70308	0.80285	0.90215	0.95147	0.99052
	0.70234	0.80449	0.90566	0.00000	0.00000
30	0.86812	0.90208	0.92592	0.94070	0.96070
	0.70624	0.80498	0.89601	0.93725	0.97199
	0.73439	0.82844	0.90974	0.94402	0.97277
	0.69947	0.79947	0.89958	0.94973	0.98996
	0.70003	0.79928	0.91029	0.00000	0.00000
40	0.87730	0.90690	0.92955	0.94371	0.96265
	0.71354	0.81251	0.90233	0.94189	0.97432
	0.74355	0.83708	0.91614	0.94853	0.97540
	0.69903	0.79894	0.89900	0.94919	0.98966
	0.69856	0.79987	0.90185	0.94935	0.00000
100	0.89805	0.91888	0.93962	0.95240	0.96892
	0.72900	0.82834	0.91546	0.95132	0.97912
	0.76300	0.85498	0.92909	0.95771	0.98118
	0.69935	0.79937	0.89949	0.94963	0.98987
	0.69911	0.79935	0.90077	0.95217	0.00000

Table 11 : Observed Coverage of P.I. Construction Procedures (5)

Population : The average delay of the first 25 customers in M/M/1 que with traffic intensity 0.9

No. of Replications : 1,000

Sample Size	Desired coverage (Confidence level)				
	0.70000	0.80000	0.90000	0.95000	0.99000
5	0.48687	0.58530	0.68473	0.76000	0.86510
	0.00000	0.00000	0.00000	0.00000	0.00000
	0.00000	0.00000	0.00000	0.00000	0.00000
	0.47523	0.57953	0.71083	0.80037	0.89983
	0.00000	0.00000	0.00000	0.00000	0.00000
10	0.53440	0.64397	0.74723	0.79833	0.86813
	0.00000	0.00000	0.00000	0.00000	0.00000
	0.43807	0.52410	0.63827	0.71427	0.81237
	0.52497	0.62733	0.74617	0.82523	0.91383
	0.53437	0.62063	0.00000	0.00000	0.00000
20	0.61580	0.72703	0.81013	0.85193	0.90330
	0.00000	0.00000	0.00000	0.00000	0.00000
	0.52293	0.62540	0.74493	0.82090	0.90597
	0.57343	0.68313	0.81180	0.88717	0.95987
	0.59573	0.69007	0.79240	0.00000	0.00000
40	0.68307	0.79153	0.85033	0.87950	0.91953
	0.57903	0.68487	0.80840	0.88157	0.95150
	0.58213	0.68910	0.81117	0.88063	0.94700
	0.60733	0.71450	0.84363	0.91673	0.97800
	0.63450	0.72863	0.82757	0.87963	0.00000
100	0.76903	0.84693	0.88487	0.90650	0.93990
	0.63137	0.74383	0.86323	0.92643	0.97493
	0.64060	0.75313	0.86933	0.92770	0.97297
	0.63940	0.75190	0.87753	0.94307	0.99207
	0.67333	0.76850	0.86470	0.91677	0.00000

Table 12 : Observed Coverage of P. I. Construction Procedures (6)

Population : Reliability model $G=\min(G_1, \max(F_2, G_3))$

with G_i ~ Weibull(0.5, 1.0)

No. of Replications : 1,000

Sample Size	Desired coverage (Confidence level)				
	0.70000	0.80000	0.90000	0.95000	0.99000
100	0.89927	0.91683	0.93524	0.94754	0.96530
	0.71097	0.80640	0.89653	0.94184	0.97724
	0.68810	0.78823	0.88877	0.94034	0.98500
	0.74620	0.84347	0.92257	0.95107	0.97607
	0.69390	0.79470	0.89617	0.94787	0.00000
40	0.89283	0.90993	0.92997	0.94337	0.96044
	0.70497	0.79810	0.89153	0.93670	0.97424
	0.68377	0.78247	0.88203	0.93567	0.98190
	0.74360	0.84070	0.91763	0.95000	0.97690
	0.70060	0.79983	0.90057	0.94897	0.00000
20	0.87327	0.89450	0.91487	0.93140	0.95107
	0.67387	0.77067	0.86753	0.91627	0.96384
	0.65510	0.75703	0.86073	0.91680	0.97124
	0.73020	0.82740	0.90883	0.94373	0.97470
	0.69520	0.79367	0.89340	0.00000	0.00000
10	0.84613	0.87447	0.89837	0.91580	0.94100
	0.64013	0.73280	0.83083	0.88660	0.94314
	0.62533	0.72290	0.82680	0.88860	0.95243
	0.72833	0.82173	0.90443	0.94227	0.97570
	0.71327	0.79833	0.00000	0.00000	0.00000
5	0.79387	0.83993	0.87510	0.89623	0.93393
	0.00000	0.00000	0.00000	0.00000	0.00000
	0.54833	0.64130	0.74673	0.81387	0.89553
	0.71170	0.80500	0.88973	0.93547	0.97997
	0.00000	0.00000	0.00000	0.00000	0.00000