J. of Basic Sciences, Cheju Nat. Univ.
9(2), 1~8, 1996

기초과학연구 제주대학교
9(2), 1~8, 1996

RIEMANNIAN CURVATURE WITH BI–INVARIANT ON SYMMETRIC SPACES

HYUN, JIN-OH

Department of Mathematics Education
Cheju National University

1. Introduction

In this paper, we have proved some theorems which assert that every compact connected Lie group is a symmetric manifold with respect to the bi-invariant metric. Using the definition of curvature operator, we will derive some results of Riemannian curvature operator and Riemannian curvature tensor.

2. Bi-invariant Riemannian metric

Let M be a C^∞ manifold, and let $\theta : R \times M \longrightarrow M$ be a C^∞ mapping satisfying the conditions

(1)$\theta(0,p) = p$ for every $p \in M$

(2)$\theta_t \circ \theta_s(p) = \theta_{t+s}(p) = \theta_s \circ \theta_t(p)$

for every $s, t \in R$ and for every $p \in M$, where $\theta_t(p) = \theta(t,p)$.

Then θ is called a C^∞ *action* or *one parameter group* of M. For each one parameter group $\theta : R \times M \longrightarrow M$ there exists a unique C^∞ vector field X, which is called the *infinitesimal generator* of θ, such that

$$X_p f = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ f\big(\theta_{\Delta t}(p)\big) - f(p) \right\}$$

for each $f \in C^\infty(p)$.

Definition 2.1. *If $\theta : G \times M \longrightarrow M$ is the action of group G on M. Then a vector field X on M is said to be invariant under each of the diffeomorphism θ_g of M to itself for every $g \in G$. That is, $\theta(g_*, X) = X$.*

Proposition 2.2. *If $\theta : R \times M \longrightarrow M$ is a C^∞ action of R on M. Then the infinitesimal generator X is invariant under this action, that is*

$$\theta_{t_*}(X_p) = X_{\theta_t(p)} \quad \text{for all } t \in R.$$

Proof. Let $f \in C^\infty(\theta_t(p))$ for some $(t,p) \in R \times M$. Then

$$\begin{aligned} \theta_{t_*}(X_p)f &= X_p(f \circ \theta_t) \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[f \circ \theta_t(\theta_{\Delta t}(p) - f \circ \theta_t(p) \right]. \end{aligned}$$

However, R is Abelian and we have

$$\theta_t \circ \theta_{\Delta t} = \theta_{t+\Delta t} = \theta_{\Delta t} \circ \theta_t.$$

So

$$\begin{aligned} \theta_{t_*}(X_p)f &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[(f \circ \theta_{\Delta t}(\theta_t(p))) - f(\theta_t(p)) \right] \\ &= X_{\theta_t(p)} f. \end{aligned}$$

Let G be a Lie group. For each $a \in G$, let La[Ra] be a left[right] transformation. That is, for every $g \in G$,

$$\begin{aligned} L_a &: G \longrightarrow G, \ L_a(g) = ag, \text{ and} \\ R_a &: G \longrightarrow G, \ R_a(g) = ga. \end{aligned}$$

If a C^∞ vector field X of G has the property that $L_{a_*}(X_g) = X_{ag}$ $(R_{a_*}(X_g) = X_{ga})$ for every $a, g \in G$, then X is said to be *left(right) invariant.*

We put $\mathcal{L} = \{X \in \mathfrak{X}(M) \mid X \text{ is a left invariant } C^\infty \text{ vector field}\}$.

Then vector space $\mathcal{L}$ is a Lie algebra with product $[X, Y]$. $\mathcal{L}$ is called the *Lie algebra* of G. In this case $\mathcal{L} \cong T_e(G)$ where e is the identity of G as Lie algebra ([5]).

Let G be a Lie group. For each $a \in G$ we define $I_a : G \longrightarrow G$ by $I_a(g) = aga^{-1}$. We can easily prove the following : For $a, b \in G$

$$L_a^{-1} = L_{a^{-1}}, \quad R_a^{-1} = R_{a^{-1}}, \quad L_a \circ R_a = R_a \circ L_a$$
$$I_a = L_a \circ R_{a^{-1}}, \quad I_{ab} = I_a \circ I_b.$$

Therefore we can get the following : For $X, Y \in \mathcal{L}$

$$\left.\begin{array}{l} (1) L_{a_*}(R_{a_*}X) = R_{a_*}(L_{a_*}X) = R_{a_*}X \in \mathcal{L} \\ (2) I_{a_*}X = L_{a_*}(R_{a^{-1}{}_*}X) = R_{a^{-1}{}_*}X \in \mathcal{L} \\ (3) I_{a_*}([X,Y]) = [I_{a_*}X, I_{a_*}Y] \in \mathcal{L} \\ (4) R_{a_*} \text{ and } I_{a_*} \text{ are automorphisms of } \mathcal{L} \end{array}\right] \cdots\cdots\cdots\cdots\cdots\cdots (*)$$

It is *bi-invariant* if it is both left and right invariant.

Definition 2.3. *Let $F : R \longrightarrow G$ be a group homomorphism, where R is a Lie group with addition and G be a Lie group. Then $F(R) = H \subset G$ is called a one parameter subgroup of G.*

Proposition 2.4. *Let G be a Lie group. Then there is an one-to-one correspondence between $\mathcal{L}$ and the set of all one-parameter subgroups of G. equally, every left invariant vector field of G is complete* ([4]).

Let Φ be a Riemannian metric on M. Then every Lie group has a left-invariant Riemannian metric and every Lie group is orientable ([4]).

From the existence of a bi-invariant volume element one is able to deduce many important properties of Lie group, if define the bilinear form Φ_e determines a bi-invariant tensor field of order 2 on $T_e(G)$. Then we have following property.

Proposition 2.5. *It is possible to defined a bi-invariant Riemannian metric Φ on a compact connected Lie group G. ([4])*

3. Symmetric Riemannian manifold

Let $\mathfrak{X}(M)$ be the set of all C^∞ vector fields over a C^∞ manifold M. Then it is obvious that $\mathfrak{X}(M)$ is a module of the commutative ring $C^\infty(M)$.

Definition 3.1. *A C^∞ connection ∇ on M is a mapping*

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

defined by $\nabla(X,Y) = \nabla_X Y$, which is satisfying conditions:
For all $f, g \in C^\infty(M)$, and $X, X', Y, Y' \in \mathfrak{X}(M)$

$$(1)\nabla_{fX+gX'}Y = f\nabla_X Y + g\nabla_{X'}Y$$
$$(2)\nabla_X(fY+gY') = f\nabla_X Y + g\nabla_X Y' + (Xf)Y + (Xg)Y'$$
$$(3)[X,Y] = \nabla_X Y - \nabla_Y X \qquad (Symmetric)$$
$$(4)X(Y,Y') = (\nabla_X Y, Y') + (Y, \nabla_X Y'),$$

where (,) is the inner product on M.
A C^∞ connection ∇ is called a Riemannian connection.

Let M be a Riemannian manifold. Then it has been proved that there exists a unique Riemannian connection ∇ ([4]).

Theorem 3.2. *Let $F : M_1 \longrightarrow M_2$ be an isometry between Riemannian manifolds M_1 and M_2. Then F preserves the Riemannian connection.*

Proof. Let $\nabla^{(1)}$ and $\nabla^{(2)}$ be Riemannian connections of M_1 and M_2, respectively. For each X', X, $Y \in \mathfrak{X}(M)$. We have

$$F_*X(F_*X', F_*Y) = X(X', Y),$$

where (,) is the inner product on the given Riemannian manifolds. In fact noting $(F_*X', F_*Y)_{F(p)} = (X', Y)_p$ for each $p \in M_1$ where F is an isometry, we have

$$\begin{aligned} F_*X(F_*X', F_*Y)_{F(p)} &= X_p(F_*X', F_*Y)_{F(p)} \\ &= X_p(F_*X', F_*Y)_{F(p)} \\ &= X_p(X', Y)_p. \end{aligned}$$

By (4) of Definition 3.1,

$$\begin{aligned} F_*(X)(F_*X', F_*Y) &= (\nabla^{(2)}_{F_*X} F_*X', F_*Y) + (F_*X', \nabla^{(2)}_{F_*X} F_*Y) \\ &= X(X', Y) \\ &= (\nabla^{(1)}_X X', Y) + (X', \nabla^{(1)}_X Y) \\ &= (F_*(\nabla^{(1)}_X X'), F_*Y) + (F_*X', F_*(\nabla^{(1)}_X Y)). \end{aligned}$$

Hence

$$\left(F_*(\nabla^{(1)}_X X') - \nabla^{(2)}_{F_*X} F_*X',\ F_*Y\right) + \left(F_*X',\ F_*(\nabla^{(1)}_X Y) - \nabla^{(2)}_{F_*X} F_*Y\right) = 0.$$

Since the above identity holds for $X,\ Y,\ X' \in \mathfrak{X}(M)$, we have

$$F_*(\nabla^{(1)}_X Y) = \nabla^{(2)}_{F_*X} F_*Y$$

for every $X,\ Y \in \mathfrak{X}(M)$.

Definition 3.3. *Let M be a connected Riemannian manifold. If to each $p \in M$ there exists an isometry $\sigma_p : M \longrightarrow M$ which is*
(1) σ_p is involutive (i.e., $\sigma_p^2 = \sigma_p$), and
(2) there exists an open neighborhood U of p such that $\sigma_p|_U$ has the only fixed point p, then M is said to be symmetric. Sometimes p is called isolated fixed point of a symmetry σ_p.

Let M be a symmetric manifold and let $\sigma_p : M \longrightarrow M$ be a symmetry at p. Then for $X_p \in T_p(M)$, $\sigma_{p_*}(X_p) = -X_p$, where p_* denoting the point antipodal at p. ([5])

Proposition 3.4. *A symmetric Riemannian manifold M is complete. Furthermore, for each $p \in M$ the symmetry σ_p at p maps a geodesic on M through p onto itself.*([5])

Theorem 3.5. *Every compact and connected Lie group G is the symmetric space with respect to the bi-invariant metric. Thus with the bi-invariant metric G is complete.*

Proof. By proposition 2.5, G has the bi-invariant metric. Define $\Psi : G \longrightarrow G$ by $\Psi(x) = x^{-1}$ for each $x \in G$. It follows that Ψ is involute because that Ψ has only one fixed point e(identity of G). Recall that for each $X_e \in T_e(G)$ there exists a unique one parameter subgroup $F : R \longrightarrow G$ such that $X_e = \dot{F}(0)$. If $x = F(t)$ then $x^{-1} = F(-t)$ and thus $\Psi(F(t)) = F(-t)$. Hence

$$\begin{aligned}\Psi_*(X_e) = \Psi_*(\dot{F}(0)) &= \frac{d}{dt}(\Psi(F(t)))\,|_{t=0} \\ &= \frac{d}{dt}F(-t)\,|_{t=0} = -\dot{F}(0) = -X_e.\end{aligned}$$

It follows that for $X_e,\ Y_e \in T_e(G)$

$$(\Psi_{*e}X_e, \Psi_{*e}Y_e) = (-X_e, -Y_e)$$
$$= (X_e, Y_e),$$

where (,) is the bi-invariant inner product on $T_e(G)$. That is, Ψ_{*e} is an isometry on $T_e(G)$. Note that L_a and $R_a (a \in G)$ are isometries with respect to the bi-invariant metric of G. Since

$$\Psi(x) = x^{-1} = (a^{-1}x)^{-1}a^{-1} = R_{a^{-1}} \cdot \Psi \cdot L_{a^{-1}}(x)$$

for each $x \in G$ $\Psi_{*a} : T_a(G) \longrightarrow T_{a^{-1}}(G)$ may be written as

$$\Psi_{*a} = (R_{a^{-1}{}_*})_e \cdot \Psi_{*e} \cdot (L_{a^{-1}{}_*})a.$$

Thus Ψ_{*a} is an isometry. In consequence, $\Psi : G \longrightarrow G$ is an isometry. For each $g \in G$ define σ_g by

$$\sigma_g = L_g \cdot R_g \cdot \Psi, \text{ that is, } \sigma_g(x) = gx^{-1}g.$$

Then it follows that σ_g is the symmetry at G.

Proposition 3.6. *Let G be a compact connected Lie group. Then each geodesic through the identity e of G is a one parameter subgroup of G. Furthermore every point of a connected Lie group G is a one parameter subgroup. Thus this geodesic is an one parameter subgroup and so G is a one parameter subgroup.*([4])

Let M be a Riemannian manifold. For C^∞ vector fields $X,\ Y$ over M, the *curvature operator* $R(X,Y)$ is defined by

$$R(X,Y) \cdot Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$$

for each C^∞ vector field Z over M, where ∇ is the Riemannian connection of M.

Theorem 3.7. *Let G be a compact connected Lie group and let $\mathcal{L}$ be the Lie algebra of G. For $X, Y, Z \in \mathcal{L}$, Riemannian curvature operator equal to $\frac{1}{4}[Z,[X,Y]]$ with bi-invariant Riemannian metric.*

Proof. Let ∇ be the Riemannian connection with bi-invariant metric of G. Take $X \in \mathcal{L}$ then $\nabla_X X = 0$. In fact, X_e define a unique one parameter subgroup $F : R \longrightarrow G$ such that $F(0) = e$ and $\dot{F}(0) = X_e$. For a C^∞ vector field Y over M,

$$\nabla_{X_e} Y = \frac{D}{dt} Y_{F(t)}|_{t=0}.$$

Hence

$$\nabla_{X_e} X = \frac{D}{dt} X_{F(t)}|_{t=0}.$$

$F(t)$ is geodesic by proposition 3.6 and thus

$$\frac{D}{dt} X_{F(t)} = \frac{D}{dt}\left(\frac{dF}{dt}\right) = 0.$$

This means that $\nabla_{X_e} X = 0$. Since our metric is left-invariant and X is also left-invariant, by Theorem 3.2 $\nabla_X X = 0$ everywhere on G. Since if X and Y are left invariant vector fields then so are $X + Y$ and $[X,Y]$, we have

$$0 = \nabla_{X+Y}(X+Y) = \nabla_X Y + \nabla_Y X \ (\nabla_X X = 0 = \nabla_Y Y).$$

If X and Y are left invariant, then

$$\nabla_X Y + \nabla_Y X = 0, \quad [X,Y] = \nabla_X Y - \nabla_Y X.$$

By properties (*), the connection of a biinvariant metric on G is given by

$$\nabla_X Y = \frac{1}{2}[X,Y].$$

For X, Y and Z in $\mathcal{L}$, since

$$\begin{aligned}
\nabla_X(\nabla_Y Z) &= \frac{1}{2}[X, \nabla_Y Z] = \frac{1}{2}[X, \frac{1}{2}[Y,Z]] \\
&= \frac{1}{4}[X,[Y,Z]] \\
\nabla_X(\nabla_X Z) &= \frac{1}{4}[Y,[X,Z]] \\
\nabla_{[X,Y]} Z &= \frac{1}{2}[[X,Y],Z]
\end{aligned}$$

we have the following;

$$\begin{aligned} R(X,Y)\cdot Z &= \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z \\ &= \frac{1}{4}[X,[Y,Z]] - \frac{1}{4}[Y,[X,Z]] - \frac{1}{2}[[X,Y],Z] \\ &= \frac{1}{4}\{[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]]\} + \frac{1}{4}[Z,[X,Y]] \\ &= \frac{1}{4}[Z,[X,Y]]. \end{aligned}$$

Theorem 3.8. *Let G be a symmetric Riemannian manifold and let $\mathcal{L}$ be the Lie algebra of G. Then for $X, Y, Z, W \in \mathcal{L}$ Riemannian curvature tensor*

$$R(X,Y,Z,W) = -\frac{1}{4}([X,Y],[Z,W])$$

with bi-invariant Riemannian metric.

Proof. From the result $R(X,Y,Z,W) = (R(X,Y)\cdot Z, W)$, using the property $([X,Y],Z) = (X,[Y,Z])$ ([6]) and Theorem 3.7, we have

$$\begin{aligned} R(X,Y,Z,W) &= -\frac{1}{4}([[X,Y],Z],W) \\ &= -\frac{1}{4}([X,Y],[Z,W]). \end{aligned}$$

REFERENCE

[1]. Daniel Martin ; *Manifold Theory an introduction for Mathematical Physicist.* Ellis Horwood Limited (1991)

[2]. K Yano & M Kon ; *Structures on Manifold.* World Scientific Publishing Co. (1984)

[3]. R. L. Bishop ; *Geometry of Manifold.* Academic Press, INC. (1964)

[4]. W. M. Boothby ; *An Introduction to Differentiable Manifolds and Riemannian Geometry.* Academic Press (1975).

[5]. W. Klingenberg ; *Riemannian Geometry.* Walter de Gruyter (1982)

[6], S. Helgason ; *Differential Geometry, Lie groups, and Symmetric spaces.* Academic Press. (1962)

[7]. F. W. Gehring, P. R. Halmos, C. C. More ; *Modern Geometry Methods and Applications (I, II).* Springer-Verlag (1984).