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General types of idempotent (0, 1)-matrices

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Abstract

In this paper, we consider the problem of characterizing idempotent matrices over the Boolean algebra. Consequently, we obtain all types of idempotent Boolean matrices. They are turned out the sums of some rectangle parts and some line parts of the given matrices.

Keywords: Idempotent matrix, cell, canonical form, dominate, frame, rectangle part, line part, (i, j)-disjoint, minimized idempotent matrix.

AMS Subject Classifications: 15A21, 15A33

1 Introduction and Preliminaries

There are many papers on the study of characterizations of matrices over several semirings([1]-[9]). Boolean matrices([3]-[7]) also have been the subject of research by many authors because of their association with nonnegative real matrices. Beasley and Pullman [4] studied on the idempotent matrices and their preservers over several semirings, and they obtained the characterizations of idempotent Boolean matrices which are the sums of 3 cells. But there are few papers on the characterizations of idempotent Boolean matrices. Song and Kang [7] considered the following questions: What are forms of idempotent Boolean matrices? So they obtained all types of idempotent Boolean matrices which are the sums of 4 cells. But they had not obtained the general types of all idempotent Boolean matrices. In general, the characterization of idempotents in abstract algebra systems is a vital problem which is crucial for the understanding the structure of these systems and in many other applications(see [8]-[9]). Even for matrices over algebraic systems that are not field this problem is far from being solved yet. The present paper is devoted to the characterization of idempotents in matrices over the Boolean algebra.

DEFINITION 1.1. The *Boolean algebra* is the set $\mathbb{B} = \{0, 1\}$ which is equipped with two binary operations, addition and multiplication. The operations are defined as usual except that $1 + 1 = 1$.

Let $\mathcal{M}_n(\mathbb{B})$ denote the set of all $n \times n$ matrices with entries in $\mathbb{B}$. The usual definitions for adding and multiplying matrices over fields are applied to Boolean matrices as well.

Throughout this paper, all matrices are $n \times n$ Boolean matrices with entries in $\mathbb{B}$. The zero matrix is denoted by 0, the identity matrix by I and the matrix with all entries equal to 1 is denoted by J.

DEFINITION 1.2. An $n \times n$ Boolean matrix with only one entry equal to 1 is called a *cell*. If the nonzero entry occurs in i^{th} row and j^{th} column, we denote this cell by E_{ij} and say that the cell is in row i and it is in column j. For $i \neq j$, we say that E_{ij} is an *off-diagonal cell*; E_{ii} is a *diagonal cell.*

DEFINITION 1.3. A *line* is a row or a column of a matrix. A set of cells is *collinear* if they are all in the same line.

The following two Lemmas are immediate consequences of the rules of matrix multiplication.

LEMMA 1.4. *For all indices i, j, u, and v, we have $E_{ij}E_{uv} = E_{iv}$ or 0 according as $j = u$ or $j \neq u$.*

LEMMA 1.5. *Suppose that C and D are two cells with $CD \neq 0$.*

(a) *If C and D are diagonal, then $C = D$.*

(b) *If C is a diagonal cell and D is not, then $CD = D$, and C and D are in the same row. If D is a diagonal cell and C is not, then $CD = C$, and C and D are in the same column.*

(c) *If C and D are off-diagonal cells, then either*

(i) *CD is an off-diagonal cell distinct from C and D with $DC = 0$ or*

(ii) *$D = C^T$, and CD and DC are distinct diagonal cells.*

2 General types of idempotent $(0,1)$-matrices

DEFINITION 2.1. A matrix E is called *idempotent* if $E^2 = E$. Otherwise, E is called *non-idempotent*.

The matrices $0, I$ and J are clearly idempotent. It follows from Lemma 1.4 that all diagonal cells are idempotent, but all off-diagonal cells are non-idempotent.

Let $A = [a_{ij}]$ be any matrix in $\mathcal{M}_n(\mathbb{B})$. Then it can be written uniquely as

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}E_{ij},$$

which is called the *canonical form* of A. Since $a_{ij} \in \{0, 1\}$, the canonical form shows that the matrix A is a sum of cells.

In this section, we give the general types of idempotent matrices in $\mathcal{M}_n(\mathbb{B})$. For this purpose, we shall analyze the structures of the sums of some cells.

DEFINITION 2.2. We say that a matrix $A = [a_{ij}]$ *dominates* a matrix $B = [b_{ij}]$ if and only if $a_{ij} = 0$ implies that $b_{ij} = 0$, and we write $A \geq B$ or $B \leq A$.

PROPOSITION 2.3. *Let A be an idempotent matrix in $\mathcal{M}_n(\mathbb{B})$. If $E_1, \ldots, E_k \leq A$ are some cells with $k \geq 2$, then the product $E_1 \cdots E_k \leq A$.*

Proof. Since A is idempotent, A is k-potent($A^k = A$). If $E_1 \cdots E_k = 0$, then $E_1 \cdots E_k \leq A$ is obvious. Assume that $E_1 \cdots E_k \neq 0$. By Lemma 1.4, $E_1 \cdots E_k$ is a cell which is a summand for the matrix A^k. By the addition rules in $\mathbb{B}$, there is no elements that can cancel a non-zero summand. Thus $E_1 \cdots E_k \leq A^k = A$. The result follows. ∎

LEMMA 2.4. *Let A be a matrix in $\mathcal{M}_n(\mathbb{B})$. Then*

(1) *If all cells of A are diagonal, then A is idempotent.*

(2) *If all cells of A are off-diagonal, then A is non-idempotent.*

Proof. (1) is obvious by Lemma 1.4. Now, we will prove (2). Suppose that all cells of A are off-diagonal, and let $\mathcal{O} = \{F_1, \ldots, F_m\}$ be the set of all off-diagonal cells in A. Thus we have $A = \sum_{i=1}^{m} F_i$. Let us show that if A is idempotent, then there exists an infinite set of cells in $\mathcal{O}$, which is impossible. We proceed by induction.

Since A is idempotent, there exist distinct three cells F_i, F_j and F_l in $\mathcal{O}$ such that $F_iF_j = F_l$. By Lemma 1.4, we can write $F_i = E_{ax_1}$, $F_j = E_{x_1b}$, and $F_l = E_{ab}$ with mutually distinct indices a, b and x_1. Since $F_i = E_{ax_1} \in \mathcal{O}$ and A is idempotent, there exist two distinct cells E_{ax_2} and $E_{x_2x_1}$ in $\mathcal{O}$ such that $E_{ax_1} = E_{ax_2}E_{x_2x_1}$ for some index x_2 different from a and x_1. Assume that for some $k \geq 2$, the set of distinct cells $\{E_{ax_1}, \ldots, E_{ax_k}\} \in \mathcal{O}$ was already constructed. Then we may add a new element to this set as follows: Since A is idempotent, there exist two distinct cells $E_{ax_{k+1}}, E_{x_{k+1}x_k} \in \mathcal{O}$ such that $E_{ax_k} = E_{ax_{k+1}}E_{x_{k+1}x_k}$ for some index x_{k+1} different from a and x_k. Let us assume that there exists an index $i = 1, \ldots, k-1$ such that $x_i = x_{k+1}$. Hence we have that

$$E_{x_ix_i} = E_{x_{k+1}x_i} = E_{x_{k+1}x_k} \cdots E_{x_{i+1}x_i} \leq A,$$

a contradiction. Thus $x_i \neq x_{k+1}$ for all $i = 1, \ldots, k$. It follows that $E_{ax_i} \in \mathcal{O}$ are distinct cells for $i = 1, \ldots, k+1$. Hence, $\mathcal{O}$ contains an infinite set of distinct cells. This contradiction concludes the proof that A is non-idempotent. ∎

DEFINITION 2.5. Let C_1, C_2, C_3 and C_4 be four distinct cells in $\mathcal{M}_n(\mathbb{B})$. Then their sum is called a *frame* if the four 1's constitute a rectangle such that at least one of them lies on the main diagonal of the matrix $\sum_{i=1}^{4} C_i$. In this case we will say that each cell C_i, $i = 1, 2, 3, 4$, is in the frame.

For example, the following two matrices A_1 and A_2 are frames, but B is not.

$$A_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (2.1)$$

In fact, we can easily show that A_1 and A_2 are idempotent, but B is not in $\mathcal{M}_4(\mathbb{B})$.

PROPOSITION 2.6. *Let A be idempotent in $\mathcal{M}_n(\mathbb{B})$. If F is an off-diagonal cell in A such that it is not in the same line to any diagonal cell in A, then it make a frame with one diagonal cell and two off-diagonal cells in A.*

Proof. Let $\mathcal{D} = \{E_1, \cdots, E_m\}$ and $\mathcal{O} = \{F_1, \cdots, F_l\}$ be the sets of all distinct diagonal and off-diagonal cells of A, respectively. Then we have that

$$A = \sum_{i=1}^{m} E_i + \sum_{j=1}^{l} F_j.$$

By Lemma 2.4, we have that $m \geq 1$. Let us denote $E_i = E_{a_i a_i}$ for all $i = 1, \ldots, m$ and $F = E_{bc}$. Since F and E_i are not collinear for all i, it follows that $a_1, \ldots, a_m, b$, and c are mutually distinct indices. We will show that there exists an index $q \in \{1, \ldots, m\}$ such that the four cells

$$E_{a_q a_q}, \quad E_{b a_q}, \quad E_{a_q c}, \quad \text{and} \quad E_{bc}$$

is a frame. If not, as in the proof of Lemma 2.4 we will construct an infinite set of cells in $\mathcal{O}$ applying the induction process.

Since A is idempotent and F is not in the same line to any cell in $\mathcal{D}$, there exist two distinct cells E_{bx_1} and E_{x_1c} in $\mathcal{O}$ such that $E_{bc} = E_{bx_1}E_{x_1c}$ for some index x_1 different from b and c. If $x_1 = a_i$ for some i, then $q = i$ leads to a contradiction. Hence, $x_1 \neq a_i$ for all i.

Since A is idempotent and $E_{bx_1} \in \mathcal{O}$, we can find two cells E_{bx_2} and $E_{x_2x_1}$ in $\mathcal{O}\cup\mathcal{D}$ such that $E_{bx_1} = E_{bx_2}E_{x_2x_1}$ for some index x_2. If $b = x_2$, then $E_{x_2x_2}$ and $F = E_{bc}$ are in the same line, a contradiction. Hence, $b \neq x_2$ so that $E_{bx_2} \in \mathcal{O}$. If $x_2 = x_1$, then the four cells $E_{x_2x_1}, E_{bx_2}, E_{x_1c}$ and $E_{bc} = F$ are in the frame, a contradiction so that $E_{x_2x_1} \in \mathcal{O}$. If $x_2 = a_i$ for some i, then $q = i$ leads to a contradiction. Thus $x_2 \neq a_i$ for all i and $x_2 \neq x_1$. Assume that for some $k \geq 2$, the set of cells

$$\{E_{bx_1}, \ldots, E_{bx_k}, E_{x_1c}, E_{x_2x_1}, \ldots, E_{x_kx_{k-1}}\} \in \mathcal{O}$$

is already constructed. Then we may add new elements to this set as follows: Since A is idempotent, there exist two cells $E_{bx_{k+1}}$ and $E_{x_{k+1}x_k}$ in $\mathcal{O} \cup \mathcal{D}$ such that $E_{bx_k} = E_{bx_{k+1}}E_{x_{k+1}x_k}$ for some index x_{k+1}. Then we have $x_{k+1} \neq b$ because F is not collinear with diagonal cells. Assume that $x_{k+1} = x_k$. Then $E_{x_{k+1}c} = E_{x_{k+1}x_k} \cdots E_{x_2x_1}E_{x_1c} \leq A$ and by considering $a_q = x_{k+1}$ we obtain a contradiction with the assumption. Thus we have

$$E_{bx_{k+1}},\ E_{x_{k+1}x_k} \in \mathcal{O}.$$

Suppose that $x_{k+1} = x_i$ for some $i = 1, \ldots, k-1$. Then $E_{x_ix_i} = E_{x_{k+1}x_i} = E_{x_{k+1}x_k} \cdots E_{x_{i+1}x_i} \leq A$. Therefore, the choice $a_q = x_i$ leads to a contradiction. Thus $x_{k+1} \neq x_i$ and we have constructed the set

$$\{E_{bx_1}, \ldots, E_{bx_{k+1}}, E_{x_1c}, E_{x_2x_1}, \ldots, E_{x_{k+1}x_k}\} \in \mathcal{O}.$$

Therefore we obtain an infinite set of off-diagonal cells, $\{E_{bx_i} | \ i \in \mathbb{N}\}$ on b^{th} row which is impossible. This contradiction concludes the proof. ∎

COROLLARY 2.7. *Let $A = F + \sum_{i=1}^{m} E_i$ be a matrix in $\mathcal{M}_n(\mathbb{B})$, where F is an off-diagonal cell and E_i diagonal cells. Then A is idempotent if and only if F is in the* *same line to at least one cell E_i for some i.*

Proof. It follows from Proposition 2.6 and Lemma 2.4. ■

COROLLARY 2.8. *Let $A = \sum_{i=1}^{m} E_i + \sum_{j=1}^{2} F_j$ be a matrix in $\mathcal{M}_n(\mathbb{B})$, where E_i are diagonal cells and F_j off-diagonal cells. Then A is idempotent if and only if each F_j is in the same line to some diagonal cell in A and it satisfies just one of the following conditions:*

(1) $F_1F_2 = F_2F_1 = 0$;

(2) *F_1 and F_2 are in a frame with two diagonal cells of A.*

Proof. Suppose that A is idempotent. By Lemma 2.4, we have that $m \geq 1$. It follows from Proposition 2.6 that each F_j is in the same line to some diagonal cell in A. Suppose that $F_1F_2 \neq 0$ or $F_2F_1 \neq 0$. If $F_1F_2 \neq 0$, then we have $F_2 = F_1^T$, and F_1F_2 and F_2F_1 are distinct diagonal cells in A by Lemma 1.5-(c). Therefore the four cells F_1, F_2, F_1F_2 and F_2F_1 form a frame. For $F_2F_1 \neq 0$, we have the same conclusion as the above. The converse is immediate. ■

Let

$$A = [a_{ij}] = \begin{bmatrix} \boldsymbol{R}_1 \\ \boldsymbol{R}_2 \\ \vdots \\ \boldsymbol{R}_n \end{bmatrix} = [\boldsymbol{C}_1 \ \boldsymbol{C}_2 \ \cdots \ \boldsymbol{C}_n]$$

be an $n \times n$ Boolean matrix, where $\boldsymbol{R}_i$ and $\boldsymbol{C}_j$ are i^{th} row and j^{th} column of A, respectively. If $a_{ij} = 1$ for some i and j, then we say that the cell E_{ij} is in the row $\boldsymbol{R}_i$ and in the column $\boldsymbol{C}_j$.

DEFINITION 2.9. Let $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{B})$. For $1 \leq i, j \leq n$, the row $\boldsymbol{R}_i$ and the column $\boldsymbol{C}_j$ are said to be *(i, j)-disjoint* if $XY = 0$ for all off-diagonal cells $X \in \boldsymbol{R}_i$ and $Y \in \boldsymbol{C}_j$.

LEMMA 2.10. *Let $A \in \mathcal{M}_n(\mathbb{B})$ be idempotent. If $\boldsymbol{R}_i$ and $\boldsymbol{C}_j$ of A are not (i,j)-disjoint, then $E_{ij} \leq A$.*

Proof. Suppose that $\boldsymbol{R}_i$ and $\boldsymbol{C}_j$ are not (i,j)-disjoint for some i, j. Then there exist at least two off-diagonal cells $X \in \boldsymbol{R}_i$ and $Y \in \boldsymbol{C}_j$ such that $XY \neq 0$. Thus we may write that $X = E_{ix}$ and $Y = E_{yj}$ for some indices x and y. Since $XY \neq 0$, it follows from Lemma 1.4 that $x = y$ and $XY = E_{ij}$. Since A is idempotent, from Proposition 2.3 it follows that $XY \leq A$, i.e., $E_{ij} \leq A$. ∎

DEFINITION 2.11. A *weight* of $A \in \mathcal{M}_n(\mathbb{B})$ is the number of non-zero entries of A and will be denoted by $|A|$.

LEMMA 2.12. *Let $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{B})$ be idempotent with $a_{ii} = 1$ for some i. If $|\boldsymbol{R}_i| = s + 1$ and $|\boldsymbol{C}_j| = t + 1$, then A has exactly $s \cdot t$ frames containing E_{ii}.*

Proof. If $s = 0$ or $t = 0$, then the proposition is straightforward. Thus we can assume that $s, t \geq 1$. Suppose that $R = \{F_1, \cdots, F_s\}$ and $C = \{G_1, \cdots, G_t\}$ are the sets of all off-diagonal cells in A which are in $\boldsymbol{R}_i$ and $\boldsymbol{C}_i$, respectively. Let F_r and G_l be arbitrary members in R and C, respectively. Then we have forms $F_r = E_{ia}$ and $G_l = E_{bi}$ for some indices a and b different from i. Since A is idempotent, $G_l F_r = E_{bi} E_{ia} = E_{ba}$ is a cell in A. Therefore the four cells

$$E_{ii}, \ F_r, \ G_l, \quad \text{and} \quad G_l F_r$$

form a frame. Thus A has at least $s \cdot t$ frames containing E_{ii}. It follows from the definition of a frame that A has at most $s \cdot t$ frames containing E_{ii}. ∎

DEFINITION 2.13. Let A be a matrix in $\mathcal{M}_n(\mathbb{B})$. We say that A has i^{th} *rectangle part* if the following holds:

(1) there is a frame in A containing E_{ii},

(2) for any $k, l \in \{1, \ldots, n\}$, if E_{li}, $E_{ik} \leq A$, then $E_{lk} \leq A$.

Let $t = |\boldsymbol{R}_i| - 1$ and $s = |\boldsymbol{C}_i| - 1$ be the numbers of non-zero off-diagonal entries in i^{th} row and j^{th} column, respectively. Then the sum

$$\sum_{k=1}^{s}\sum_{l=1}^{t}(E_{ii} + E_{ii_l} + E_{j_k i} + E_{j_k i_l}) \tag{2.2}$$

is called the i^{th} *rectangle part* of A, and is denoted by $RP(i)$.

DEFINITION 2.14. A matrix $A = [a_{ij}]$ in $\mathcal{M}_n(\mathbb{B})$ has an i^{th} *line part* if $a_{ii} = 1$ and, $|\boldsymbol{R}_i| = 1$ or $|\boldsymbol{C}_i| = 1$. In this case $\boldsymbol{R}_i + \boldsymbol{C}_i$ is a line and is called the i^{th} *line part* of A, and is denoted by $L(i)$.

COROLLARY 2.15. *If A is an idempotent matrix in $\mathcal{M}_n(\mathbb{B})$, then A is a sum of rectangle parts and line parts of A.*

Proof. It follows directly from Proposition 2.6 and Lemma 2.12. ∎

But the following Example shows that the converse of Corollary 2.15 is not true.

EXAMPLE 2.16. Let

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

be a matrix in $\mathcal{M}_4(\mathbb{B})$. Then C is the sum of the 1^{st} rectangle part and the 4 line part of C. Notice that $\boldsymbol{R}_1$ and $\boldsymbol{C}_4$ are not $(1,4)$-disjoint because $E_{13}E_{34}(= E_{14}) \neq 0$. Lemma 2.10 implies that C is not idempotent. ∎

EXAMPLE 2.17. Consider a matrix

$$D = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \in \mathcal{M}_5(\mathbb{B}).$$

Then we can easily show that 1^{st}, 2^{nd} and 4^{th} rectangle parts of D are identical. Also Theorem 2.20(below) shows that D is idempotent. ■

DEFINITION 2.18. Let $A = [a_{ij}]$ be an idempotent matrix in $\mathcal{M}_n(\mathbb{B})$. Suppose that A has i^{th} and j^{th} rectangle parts $RP(i)$ and $RP(j)$ of A with $i \neq j$. It is said that $RP(i)$ and $RP(j)$ are *disjoint* if $\boldsymbol{R_i}$ and $\boldsymbol{C_j}$ are (i,j)-disjoint or $\boldsymbol{R_j}$ and $\boldsymbol{C_i}$ are (j,i)-disjoint.

PROPOSITION 2.19. *Let $A = [a_{ij}]$ be an idempotent matrix in $\mathcal{M}_n(\mathbb{B})$. Then any two rectangle parts of A are either disjoint or identical.*

Proof. Suppose that i^{th} and j^{th} rectangle parts of A are not disjoint. By the definition, we have $\boldsymbol{R_i}$ and $\boldsymbol{C_j}$ are not (i,j)-disjoint and, $\boldsymbol{R_j}$ and $\boldsymbol{C_i}$ are not (j,i)-disjoint. Therefore E_{ij} and E_{ji} are off-diagonal cells in A by Lemma 2.10. We claim that E is a cell which is in $RP(i)$ if and only if it is a cell in $RP(j)$. It is easy to show that the four cells E_{ii}, E_{jj}, E_{ij} and E_{ji} are in $RP(i) \cap RP(j)$. Suppose that E is a cell in $RP(i)$. First, assume that $E = E_{ia}$ is an off-diagonal cell in $\boldsymbol{R_i}$. Then we have $E_{ja} = E_{ji}E_{ia} \leq A$, and the four cells

$$E_{ia},\ E_{ij},\ E_{ja} \quad \text{and} \quad E_{jj}$$

form a frame. Therefore, $E = E_{ia}$ is in $RP(j)$. Similarly, if $E = E_{bi}$ is an off-diagonal cell in $\boldsymbol{C_i}$, we obtain that $E = E_{bi}$ is in $RP(j)$.

Next, assume that $E = E_{cd}$ is an off-diagonal cell which is in neither $\boldsymbol{R_i}$ nor $\boldsymbol{C_i}$. Since E is in $RP(i)$, there exist two off-diagonal cells E_{ix} and E_{yi} in $\boldsymbol{R_i}$ and $\boldsymbol{C_i}$, respectively such that $E_{cd} = E_{yi}E_{ix}$. Therefore we have that $c = y$ and $d = x$ by Lemma 1.4. Since A is idempotent, we obtain that

$$E_{cj} = E_{yj} = E_{yi}E_{ij} \leq A \quad \text{and} \quad E_{jd} = E_{jx} = E_{ji}E_{ix} \leq A.$$

Hence the four cells

$$E_{cd},\ E_{cj},\ E_{jd} \quad \text{and} \quad E_{jj}$$

form a frame. Therefore we have that $E = E_{cd}$ is in $RP(j)$.

Similarly, if E is a cell in $RP(j)$, then we have that E is in $RP(i)$. Therefore, the two rectangle parts $RP(i)$ and $RP(j)$ are identical. ∎

THEOREM 2.20. *Let*

$$A = \sum_{i=1}^{m} E_i + \sum_{j=1}^{k} F_j$$

be a non-zero matrix in $\mathcal{M}_n(\mathbb{B})$*, where* E_i *are diagonal cells, and* F_j *off-diagonal cells. Then* A *is idempotent if and only if it is a sum of* $s(\geq 0)$ *disjoint rectangle parts and* $t(\geq 0)$ *line parts of* A*, and the following conditions are satisfied;*

(1) *if each rectangle part has* α_i *distinct diagonal cells for* $i = 1, \cdots, s$*, we have* $m = \alpha_1 + \cdots + \alpha_s + t$*,*

(2) *if* $\boldsymbol{R_i}$ *and* $\boldsymbol{C_j}$ *are not* (i, j)*-disjoint, we have* $E_{ij} \leq A$*.*

Proof. The necessity is immediate. So, we only prove the sufficiency. By Lemma 2.4, we have that $m \geq 1$. Let F be an off-diagonal cell in A. By Proposition 2.6, F is in some rectangle part or some line part of A. Therefore without loss of generality, we can assume that A has s disjoint rectangle parts and t line parts, where $s, t \geq 0$. The rests of Theorem follow from Lemma 2.10 and Proposition 2.19. ∎

COROLLARY 2.21. *Let* $A = E_{ii} + \sum_{j=1}^{k} F_j$ *be a matrix in* $\mathcal{M}_n(\mathbb{B})$*, where* E_{ii} *is a diagonal cell and* F_j *off-diagonal cells. Then* A *is idempotent if and only if one of the following conditions is satisfied;*

(1) A *is the* i^{th} *line part of* A *(i.e., all cells in* A *are collinear),*

(2) A *is the* i^{th} *rectangle part of* A*. Furthermore, if* $\boldsymbol{R_i}$ *and* $\boldsymbol{C_i}$ *have* x *and* y *off-diagonal cells, respectively, then* $k = xy + x + y$*.*

Proof. This is a special case of Theorem 2.20 with $m = 1$. The formula $k = xy + x + y$ is established by Lemma 2.12 because A has only one diagonal cell. ∎

Thus we have characterizations of all types of idempotent Boolean matrices in $\mathcal{M}_n(\mathbb{B})$ as shown in Theorem 2.20.

3 A minimized idempotent matrices

DEFINITION 3.1. For a matrix $X \in \mathcal{M}_n(\mathbb{B})$, a *minimized idempotent matrix* of X is a matrix $\overline{X}$ such that

(1) $X \leq \overline{X}$;

(2) $\overline{X}$ is idempotent;

(3) $|\overline{X}| = \min\{|Y| : X \leq Y,\ Y \text{ is idempotent}\}$.

In particular, if X is an idempotent matrix, then $\overline{X} = X$. Also, a minimized idempotent matrix of X may not be unique. For example, see the following Example.

EXAMPLE 3.2. Let

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{3.1}$$

be a matrix in $\mathcal{M}_4(\mathbb{B})$. Then $H = E_{11} + E_{22} + E_{43}$ and E_{43} is not in the same line to any diagonal cell of H. By Proposition 2.6, the off-diagonal cell E_{43} is in a frame or in a line part of $\overline{H}$. Two possibilities exist and they are

$$\overline{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{or} \quad \overline{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \tag{3.2}$$

■

Theorem 2.20 is a key to find a minimized idempotent matrix of the given matrix.

EXAMPLE 3.3. Let

$$X = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}. \tag{3.3}$$

be a matrix in $\mathcal{M}_7(\mathbb{B})$. Then X is the sum of three diagonal cells E_{11}, E_{66}, E_{77} and six off-diagonal cells $E_{14}, E_{15}, E_{21}, E_{31}, E_{56}, E_{74}$. And it is easy to show that X is not idempotent. To obtain a minimized idempotent matrix of X, it must have more cells. To do this, we use Theorem 2.20. Notice that $\boldsymbol{R}_1$ and $\boldsymbol{C}_6$ are not $(1,6)$-disjoint. Thus we have $E_{16} \leq \overline{A}$. Similarly, $E_{71} \leq \overline{X}$. Therefore the 1^{th} row of $\overline{X}$ has three off-diagonal cells E_{14}, E_{15}, E_{16}, and the 1^{th} column of $\overline{X}$ has three off-diagonal cells E_{21}, E_{31}, E_{71}. Thus $\overline{X}$ has 1^{th} rectangle part. Consequently, we obtain a minimized idempotent matrix of X as following;

$$\overline{X} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}. \tag{3.4}$$

Then we have that $\overline{X}$ is the sum of one rectangle part and two line parts. ∎

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