On Rarely Continuous Functions

Song Seok-zun

Rarely 連續函數에 關한 硏究

宋 錫 準

1. Introduction and Preliminaries

Professor V. Popa [6] has used the concept of a rare set (a set R is *rare* if Int $(R)=\emptyset$) to define a rarely continuous function as follows:

Definition 1.1 ([Long and Herrington], [popa]) A function $f: X \to Y$ is *rarely continuous* at x in X if for each open V containing f(x) there exists a rare set R_v with Cl $(R_v) \cap V = \emptyset$ and an open U containing x such that $f(U) \subset V \cup R_v$. And f is called rarely continuous if it is so at each x in X.

The purpose of this paper is to further investigate fundamental properties of such functions. Rarely continuous functions are a natural extension of weakly continuous functions.

師範大學 助敎授

Definition 1.2 ([Taqdir Husain]) (1) A function $f: X \to Y$ is *weakly continuous* at x in X if for each open V containing f(x) there exists an open U containing x such that $f(U) \subset Cl(V)$. (2) A function $f: X \to Y$ is said to be *nearly continuous* at x in X if for each neighborhood V of f(x) there is a neighborhood U of x such that $f(U) \subset Int(Cl(V))$.

Evidently, every weakly continuous function is rarely continuous but the converse is not true ([Long and Herrington]).

Definition 1.3 ([Takashi Noiri, 1978]) A subset S of a space X is said to be *N-closed* to X if for every cover $\{U_\alpha : \alpha \in \nabla\}$ of S by open sets of X, there exists a finite subfamily ∇_0 of ∇ such that

$S \subset \{Int(Cl(U_\alpha)) : \alpha \in \nabla_0\}$.
If X is N-closed relative to X, then it is called *nearly-compact.*

We refer the other definitions to [Taqdir Husain].

Lemma 1.4 ([Long and Herrington, theoreml]) Let $f: X \to Y$ be a function. Then the followings are equivalent:

(a) f is rarely continuous at x in X.

(b) For each open V containing f(x) there exists a rare set R_v with $R_v \cap Cl(V) = \emptyset$ and an open U containing x such that $f(U) \subset Cl(V) \cup R_v$.

(c) For each regular-open V containing f(x) there exists a rare set R_v with $Cl(R_v) \cap V = \emptyset$ and an open U containing x such that $f(U) \subset V \cup R_v$.

2. Properties on Rarely Continuous Functions

Theorem 2.1. A function $f: X \to Y$ is rarely continuous if and only if for each open set V in Y, there exists a rare set R_v such that

$f^{-1}(V) \subset Int(f^{-1}(V \cup R_v))$.

Proof. If f is rarely continuous, then for each open V of Y with $f(x) \in V$, there exists a rare set R_v such that $Cl(V) \cap R_v = \emptyset$ and there exists an open set U containing x such that $f(U) \subset V \cup R_v$, which implies $U \subset f^{-1}(V \cup R_v)$ and hence

$x, \in U \subset Int(f^{-1}(V \cup R_v))$.

Since $x \in f^{-1}(V)$,

$f^{-1}(V) \subset Int(f^{-1}(V \cup R_v))$.

Conversely, if for each open set $V \subset Y$, there exists a rare set R_v with $Cl(V) \cap R_v = \emptyset$ such that

$f^{-1}(V) \subset Int\ f^{-1}(V \cup R_v)$.

Then by putting

$U = Int(f^{-1}(V \cup R_v))$,

we see that

$$f(x) \in f(U) = f(Int\ f^{-1}(V \cup R_v)) \subset ff^{-1}(V \cup R_v) \subset V \cup R_v.$$

Hence f is rarely continuous by Lemma 1.4 (b).

Theorem 2.2 Let $\{P_\alpha\}$ be an open covering of a topological space X, Y a topological space, and f a function of X into Y. If for each α, the restriction $f \mid P_\alpha : P_\alpha \to Y$ is rarely continuous then f is rarely continuous.

Proof. Let $x \in X$. Then there exists α such that $x \in P_\alpha$. Let V be a regular-open subset of Y containing f(x). Since the restriction $f \mid P_\alpha = f_\alpha$ is rarely continuous, there exists a rare set R_v with $Cl(R_v) \cap V = \emptyset$ and an open U in P_α containing x such that $f(U) \subset V \cup R_v$. But then there is an open subset W of X such that

$x \in U = W \cap P_\alpha$.

But since U is an open set in X (because W and P_α are open in X), it follows that $f: X \to Y$ is rarely continuous by lemma 1.4(c).

Theorem 2.3. If $f: X \to Y$ is an almost continuous function and

$Cl(f^{-1}(V)) \subset f^{-1}(V \cup R_v)$

for each open V with rare set R_v such that $Cl(R_v) \cap V = \emptyset$ then f is rarely continuous.

Proof. For any point x in X and any open set $V \subset Y$ containing f(x), by the hypothesis we have

$Cl(f^{-1}(V)) \subset f^{-1}(V \cup R_v)$.

Since f is almost continuous, there exists an open U in X such that

$x \in U \subset Cl(f^{-1}(V))$.

Therefore $f(U) \subset V \cup R_v$. That is, f is rarely continuous.

Proposition 2.4. An open rarely continuous function of a topological space X into a regular space Y is continuous.

Proof. Let $f: X \to Y$ be an open rarely continuous function. Let V be an open neighborhood of f(x) for x in X. Since Y is regular, there is an open neighborhood W of f(x) such that $W \subset \overline{W} \subset V$. Since f is rarely continuous, there is a rare set R_w with $Cl(R_w) \cap W = \emptyset$ and an open neighborhood U of x such that

$f(x) \in f(U) \subset W \cup R_w$.

Since f is open function, f(U) is open set and so

$f(U) \subset Int(W \cup R_w) = W \subset V$.

Hence f is continuous.

Corollary 2.5. Let $f: X \to Y$ be an open function. Then the followings are equivalent;

(a) f is rarely continuous

(b) f is weakly continuous

(c) f is nearly continuous.

Proof. Proposition 2.4 shows that (a) implies (b). Since we have known that (b) implies (c) by [Taqdir Husain, §49 proposition 27], it suffices to show that (c) implies (a). Assume that f is nearly continuous. Then for any x in X and any open V in Y containing f(x) there exists an open neighborhood U of x in X such that $f(U) \subset Int(Cl(V))$. Let

$R_v = Int(Cl(V)) - V$.

Then R_v is a reare set with $Cl(R_v) \cap V = \emptyset$. Hence

$f(U) \subset Int(Cl(V)) = V \cup R_v$.

Therefore f is rarely continuous.

Proposition 2.6. Let $p: X \to Y$ be a quotient function. Let Z be topological space and let $g: X \to Z$ be a rarely continuous function that is constant on each set $p^{-1}(\{y\})$, for y in Y. Then g induces a rarely continuous function $f: Y \to Z$ such that $fop = g$.

Proof. For each y in Y, the set $g(p^{-1}(\{y\}))$ is an one-point set in Z (since g is constant on $p^{-1}(\{y\})$). If we let f(y) denote this point, then we have defined a function $f: Y \to Z$ such that for each $x \in X$, $f(p(x)) = g(x)$. To show that f is rarely continuous, let V be an open set in Z. Since g is rarely continuous, there exists a rare set R_v in Z and open U in X such that $g(U) \subset V \cup R_v$. Since p is a quotient function, p(U) is open in Y. Hence

$f(p(U)) = g(U) \subset V \cup R_v$.

Therefore f is rarely continuous.

Lemma 2.7. Let $f: X \to Y$ be a function. The following statements are equivalent.

(a) f is open rarely continuous

(b) f is nearly continuous

(c) the inverse image of a regular–open subset of Y is an open set in X.

(d) for each open subset V,

$f^{-1}(V) \subset Int[f^{-1}(Int(Cl(V)))]$.

Proof. Corollary 2.6 implies the equivalent of (a) and (b). And (b), (c) and (d) are equivalent by [Taqdir Husain, Theorem 11 in §49].

Theorem 2.8. Let X, Y and Z be spaces, A be a compact subset of X and B be a compact subset of Y, $f: X \times Y \to Z$ be an open rarely continuous function and W be an regular–open subset of Z containing $f(A \times B)$. Then there exists an open set U in X and an open set V in Y such that.

$A \subset U$, $B \subset V$ and $f(U \times V) \subset W$.

Proof. Since f is open rarely continuous, $f^{-1}(W)$ is open set in XxY containing AxB. For each (x,y) in AxB, there exist open sets M in X and N in Y such that $x \in M$, $y \in N$ and $M \times N \subset f^{-1}(W)$. since B is compact, for a fixed $x \in A$, there are open sets $M_1, \cdots, M_n$ in X containing x and corresponding open sets $N_1, \cdots, N_n$ in Y such that

$$B \subset Q = N_1 \cup \cdots\cdots \cup N_n$$

Let

$$P = M_1 \cap \cdots\cdots \cap M_n.$$

Then P is open in X, Q is open in Y, $x \in P$, $B \subset Q$, and $PxQ \subset f^{-1}(W)$. Since A is compact, there exist open sets $P_1, \cdots\cdots, P_m$ in X and corresponding $Q_1, \cdots\cdots, Q_m$ open in Y such that

$$B \subset V = Q_1 \cap \cdots\cdots \cap Q_m$$

and

$$A \subset U = P_1 \cup \cdots\cdots \cup P_m.$$

It follows that U and V are the required open sets.

In theorem 2.8, if X (or Y) is locally compact, then U (or V respectively) can be chosen so that Cl(U) (or Cl(V) respectively) is compact.

Theorem 2.9. Let $f: X \to Y$ be an open rarely continuous surjective function and X be a compact space. Then

(a) Y is an N-closed space

(b) Y is nearly-compact.

Proof. (a) Let $\{V_\alpha : \alpha \in \nabla\}$ be any open cover of Y.

Then

$$f^{-1}(V_\alpha) \subset Int[f^{-1}(Int(Cl(V_\alpha)))]$$

for each $\alpha \in \nabla$ by lemma 2.7 (d) and

$$X = \bigcup_\alpha f^{-1}(V_\alpha) \subset \bigcup_\alpha Int[f^{-1}(Int(Cl(V_\alpha)))].$$

Since X is compact, we have that

$$X = \bigcup_{i=1}^{n} f^{-1}(Int(Cl(V_{\alpha_i}))).$$

for some finite α_i. Then

$$Y = f(X) = f(\bigcup_{i=1}^{n} f^{-1}(Int(Cl(V_{\alpha_i}))) \subset \bigcup_{i=1}^{n} Int(Cl(V_{\alpha_i}))$$

Hence Y is N-closed.

(b) Let $\{V_\alpha : \alpha \in \nabla\}$ be any regular-open cover of Y. Then $f^{-1}(V_\alpha)$ is open in X for any $\alpha \in \nabla$ by lemma 2.7 (c), and $X = \bigcup_\alpha f^{-1}(V_\alpha)$. Since X is compact, we have that $\bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})$ covers X for some finite α_i. Then

$$Y = f(X) = \bigcup_{i=1}^{n} V_{\alpha_i}.$$

Hence Y is nearly-compact.

Literature Cited

Asha Mathur, 1979. A note on S-closed spaces. *Proc. Amer. Math. Soc.* 74(2). 350−352.

Long P. E. and Herrington L. L. 1982. Properties of rarely continuous functions. *Glasnik Matematicki* 17(37), 147−153.

Takashi Noiri, 1978. A generalization of perfect functions. *J. London Math. Soc.* 17(2), 540−544.

Takashi Noiri, 1974. On weakly continuous mappings. *Proc. Amer. Math. Soc.* 46(1), 120−124.

Taqdir Husain, 1977. Topology and Maps, Chapter 7. Plenum Press. New York and London.

Popa V. 1979. Sur certaines decomposition de la continute dans les espaces topologiques, *Glasnik Mat. Ser* Ⅲ 14(34), 359−362.

國 文 抄 錄

本 論文은 Popa 교수가 定義한 Rarely 連續函數의 性質을 硏究하여 連續函數의 性質들을 一般化하였다. 더우기, Rarely 연속함수의 새로운 同値 條件을 찾았고, 다른 弱連續 函數들과의 관계성을 調査하였으며 Rarely 連續函數에 의한 N−closed 空間과 nearly−compact 空間의 保存性에 대한 定理를 얻었다.