A Note on the Vector Valued Measures

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Vector Valued Measure들에 관한 小考

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요 약

本 論文에서는 測度(Measure) μ가 σ-algebra Σ 上의 Vector 값을 갖는 測度일 때, $L^1(\mu)$로부터 Banach 空間 B로 가는 有界 線型 演算子(Bounded Linear Operator) T가 Compact가 될 諸條件들을 調査하는데 그 目的을 두었다.

첫째로 T가 Compact가 되기 위하여서 (X, Σ, μ)가 有限正測度 空間(Finite Positive Measure Space)이라는 條件을 Atomfree Positive Measure Space로 代置할 수 있음을 보였다.

둘째로 主定理를 證明함에 있어서 簡單한 方法을 도입하였다.

I. INTRODUCTION

Liapounoff [7], in 1940, proved that the range of a countably additive bounded measure with values in a finite dimensional vector space is compact and, when the measure is atomfree, is convex. The next step was taken by Halmos [5] who in 1948 gave simplified proofs of Liapounoff's results, various versions of Liapounoff's theorem applied in the 1950's and 1960's and in 1966, Lindenstrauss [4] shortened the proof of the Liapounoff's theorem drastically. In 1968 Olech [6] investigated that the case of an unbounded measure with range in a finite dimensional vector space. And in 1969 Uhl [2] showed that the closure of the range of a vector valued measure of bounded variation with values in a Banach space, which is either reflexive space or a seperable dual space is compact, moreover, if the measure is atomfree the range is convex. In 1973 T. Cho and A. Tong [1] give a necessary and sufficient condition in order that a bounded linear mapping $L^1(\mu)$ into a Banach space be compact where (X, Σ, μ) was a finite positive measure space.

The purpose of this paper is to demonstrate that a bounded linear mapping $L^1(\mu)$ into a Banach space is compact where (X, Σ, μ) is an atomfree positive measure space instead of a finite positive measure space. And also in **Theorem 3** we give a slightly short proof of **Theorem 1** of [1].

II. THEOREMS AND LEMMAS

Let Σ be a σ-algebra of sets. By a vector valued measure we mean a countably additive set function μ on Σ whose values in a topological vector space. A set $E \in \Sigma$, is an atom of μ if $\mu(E) \neq 0$ and $E' \in \Sigma$, $E' \subset E$ imply $\mu(E') = 0$ or $\mu(E') = \mu(E)$. μ is atomfree if μ has no atoms. We begin with the following **lemma**.

LEMMA 1. Let (X, Σ, μ) be an atomfree positive measure space and let T: $L^1(\mu) \longrightarrow$ B be a bounded linear operator where B is a Banach space. For each real number c define

$R(c)=\{T(\chi_M/\mu(M)):\ M\in\Sigma,\ 0<\mu(M)<c\}$

where χ_M is the characteristic function of M. Then, $R(b)$ is a precompact set if and only if there is a real number a with $0<a<b$ such that $R(a)$ is precompact.

PROOF. Suppose that there is an a with $0<a<b$ such that $R(a)$ is precompact. Let $y\in R(b)$, i.e. $y=T(\chi_M/\mu(M))$ for some $M\in\Sigma$ with $0<\mu(M)<b$. Since μ is atomfree, there is a disjoint decomposition $\{M_1, \ldots, M_n\}$ of M where $M_i\in\Sigma$ and $0<\mu(M_i)<a$ $(i=1, 2, \ldots, n)$.

Hence $y=T(\chi_M/\mu(M))=T(\sum_{i=1}^{n}(\chi_{M_i}/\mu(M))$

$=\sum_{i=1}^{n}(\mu(M_i)/\mu(M))\ T(\chi_{M_i}/\mu(M_i))$ by linearity of T

$\in C.H.(R(a))$, the convex hull of $R(a)$, since $0\leq\mu(M_i)/\mu(M)\leq 1$, $\sum_{i=1}^{n}(\mu(M_i)/\mu(M))=1$ and $0<\mu(M_i)<a$.

Therefore the closure of $R(b)$ is a subset of the set cl $[C.H.\ (R(a))]$, the closed convex hull of the set $R(a)$, and $R(b)$ is precompact since cl $[C.H.\ (R(a))]$ is compact.

For $a<b$ the convex hull of $R(a)$ contains the union of all $R(b)$. This is an immediate consequence of the **lemma 1**.

LEMMA 2. For some $a>0$ the image of the positive functions of the unit ball of $L^1(\mu)$ is the convex hull of $R(a)$.

PROOF. If $\|f\|_1=1$ and $f>0$, then for a given $\varepsilon>0$ there is a simple function $\sum_{i=1}^{n}k_i\chi_{M_i}$ with $k_i>0$ and $0<\mu(M_i)<a$ such that $\|\sum_{i=1}^{n}k_i\chi_{M_i}\|_1=1$ and $\|f-\sum_{i=1}^{n}k_i\chi_{M_i}\|<\varepsilon$ since μ is atomfree $(i=1, 2, \ldots, n)$.

Now $T(\sum_{i=1}^{n}k_i\chi_{M_i})=\sum_{i=1}^{n}k_i\ \mu(M_i)T(\chi_{M_i}/\mu(M_i))\in C.H.\ (R(a))$

since $\sum_{i=1}^{n}k_i\ \mu(M_i)=\int(\sum_{i=1}^{n}k_i\chi_{M_i})\,d\mu$ $=\|\sum_{i=1}^{n}k_i\chi_{M_i}\|_1=1$ and $0<\mu(M_i)<a$.

Here we give a slightly short proof of **Theorem 1** of [1]

THEOREM 3. Let (X,Σ,μ) be an atomfree positive measure space and let B be a Banach space. Then a bounded linear operator T: $L^1(\mu)\longrightarrow$ B is compact if and only if the set $\{T(\chi_{M_i}/\mu(M_i)):\ M_i\in\Sigma,\ \mu(M_i)>0\}$ is precompact.

PROOF. To prove T is compact it is enough to show that there is a positive number a such that $R(a)$ is precompact by **lemma 1.** Suppose the contrary, i.e., none of $R(a)$ can be covered by a finite number of ε-balls.

$B_\varepsilon(y_i)=\{y\in B: \|y-y_i\|<\varepsilon\}$

where $B_\varepsilon(y_i)$ is the ε-ball with the center at y_i.

Let $y_1\in R(a)$.

$y_n\in R(a/n)-\cup_{i=1}^{n-1}B_\varepsilon(y_i)$ by induction.

Then $\{y_n\}$ is an infinite sequence and each y_i is apart at least the distance of ε, and so has no convergent subsequence. Since $y_n\in R(a/n)$ there is a measurable set M_n such that

$y_n=T(\chi_{M_n}/\mu(M_n))$, $(n=1, 2, \ldots)$

and $\mu(M_n)<a/n$. Choose a subsequence $\{\mu(M_{n(i)})\}$ of $\{\mu(M_n)\}$ such that

(1) $\mu(M_{n(i+1)})<(1/2^i)\ \mu(M_{n(i)})$ $(i=1,2,\ldots)$

Let $N_i=M_{n(i)}-M_{n(i+1)}$. Then

(2) $\mu(N_i)\geq\mu(M_{n(i)})-\mu(M_{n(i+1)})$

$>\mu(M_{n(i)})-(1/2^i)\ \mu(M_{n(i)})$

$=(1-2^{-i})\ \mu(M_{n(i)})>0$ $(i=1,2,\ldots)$.

Now

$$\| T(\chi_{N_i}/\mu(N_i)) - T(\chi_{M_{n(i)}}/\mu(M_{n(i)}))\|$$

$$\leq \|T\| \, \| \frac{\chi_{N_i}}{\mu(N_i)} - \frac{\chi_{M_{n(i)}}}{\mu(M_{n(i)})} \|$$

$$\leq \|T\| \, \{\mu(N_i)(\frac{1}{\mu(N_i)} - \frac{1}{\mu(M_{n(i)})} + \frac{\mu(M_{n(i+1)})}{\mu(M_{n(i)})}\}$$

$$< \|T\| \, \{1-(1-2^{-i})+2^{-i}\} \quad \text{by (1) and (2)}$$

$$= 2^{1-i} \|T\| \longrightarrow 0 \quad \text{as } i \longrightarrow \infty.$$

Thus $\{T(\chi_{M_{n(i)}}/\mu(M_{n(i)}))\}$ has a convergent subsequence. This contradicts the hypothesis, so there is a positive number a such that $R(a)$ is precompact.

REMARK. If the measure space (X, Σ, μ) is finite, the space need not be atomfree in order that the **Theorem** hold.

References

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