The Third Solutions of Semilinear Elliptic Problems

*Ko Bong-soo**

반선형인 타원형 문제에서 중간해의 존재성

高 鳳 秀*

Summary

We study multiple solutions of Semilinear Elliptic Boundary Value Problems $\Delta u(x)+f(x, u(x))=0$, $x\in\Omega$, $Bu(x)=\phi(x)$, $x\in\partial\Omega$. As one application of the result, we show that the existence of several ordered positive solutions of singularly perturbed semilinear elliptic boundary value problems as well as to the formation of bounary layer, non-uniform interior layer.

1. Introduction

In this paper we study multiple solutions of elliptic problems of the type :

$$\begin{cases} \Delta u(x) + f(x,u(x)) = 0, & x \in \Omega \\ Bu(x) = \varphi(x), & x \in \partial\Omega, \end{cases} \qquad (1)$$

where Ω is a smooth bounded open domain in R^n, $n \geq 1$, and $\partial\Omega \in C^{2,\alpha}(0<\alpha<1)$. $\triangle$ is the Laplacian operator and

$$Bu(x) = p(x)u(x) + q(x)\frac{du(x)}{d\nu},$$

where $\frac{du}{d\nu}$ denotes the outward normal derivative of u on $\partial\Omega$.

Suppose now, that $\bar{v}$, $\hat{v}$ and $\bar{w}$, $\hat{w}$ are pairs of C^2-subsolutions and C^2-supersolutions of (1) such that $\bar{v}(x) \leq \hat{w}(x)$, $\bar{w}(x) \leq \hat{w}(x)$, and $\bar{v}(x) \leq \hat{v}(x)$, for all $x\in\bar{\Omega}$ and $\bar{w}(x_0) \geq \hat{v}(x_0)$ for some $x_0\in\bar{\Omega}$. Then there is a solution in the ordered interval $[\bar{v}, \hat{v}]=\{u\in C(\bar{\Omega}) : \bar{v}(x) \leq u(x) \leq \hat{v}(x),\ x\in\bar{\Omega}\}$ and a solution in $[\bar{w}, \hat{w}]$. And furthemore it is known that there exists a third solution in the set $[\bar{v}, \hat{w}]/[\bar{v}, \hat{v}] \cup [\bar{w}, \hat{w}]$ under additional conditions [Amann].

The existence of a solution given a pair of quasi-subsolution and quasi-supersolution, $\bar{v}$ and $\hat{v}$, with $\bar{v}(x) \leq \hat{v}(x)$ for all $x\in\bar{\Omega}$, is well known [Schmitt]. Since these functions may have singular points in the interior of Ω, there arises the question, does there also exist a third solution if there are pairs of quasi-subsolutions and quasi-supersolutions as in the preceding paragraph? The author is able to prove *this multiplicity result* for the (1) using the maximum principles and the theorem on

* 사범대학 수학교육과

existence of several fixed points [pp.241, Deimling]. The result is a generalization of the one of Amann in the case f is independent of ∇u.

As one application of the result, the existence of ***several ordered positive solutions*** of singularly perturbed semi-linear elliptic boundary value problems :

$$\begin{cases} \epsilon^2\Delta u + f(x,u) = 0, & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega \end{cases}$$

is studied under some restrictions of the function f. The result is a complete answer of the open problem of de Figuerido.

Throughout this paper we assume that $p, q \in C^{1,\alpha}(\partial\Omega)$ are nonnegative real-valued functions which do not vanish simultaneously, and conditions : $0<\alpha<1$,

(i) $f \in C^{\alpha}(\bar{\Omega} \times I)$, where I is fixed finite closed interval in R^1.

(ii) There is a positive constant M such that

$$|f(x,u) - f(x,v)| < M|u-v|$$

for all $x \in \bar{\Omega}$ and $u, v \in I$.

We define a classical solution of the (1) to be a real-valued function $u : \bar{\Omega} \to I$ such that $u \in C^2(\bar{\Omega})$ and u satisfies (1) pointwise. And we define the following notation :

$$C^{\circ}(\bar{\Omega} : I) = \{u \in C^{\circ}(\bar{\Omega}) : u(x) \in I, x \in \bar{\Omega}\}$$

2. Main Results

The following results are well known.

Theorem 1 [pp 137, Ladyzhenskaya]. *For every $g \in C^{\alpha}(\bar{\Omega})$ and $\varphi \in C^{1,\alpha}(\bar{\Omega})$, the boundary value problem* (2) :

$$\begin{cases} \Delta u(x) - \lambda u(x) = g(x), & x \in \Omega \\ Bu(x) = \varphi(x), & x \in \partial\Omega, \end{cases} \tag{2}$$

where λ is a real parameter, has a uniqe solutionin $C^{2,\alpha}(\bar{\Omega})$ for all λ except for a countable number of values λ_1, $\lambda_2, \cdots$ that constitute the spectrum of (1).

In (2), if λ is not in the spectrum of (1), we can get a priori-estimate for (2).

Theorem 2 [Agmon, Amann]. *There exists a constant D such that for every $u \in C^2(\bar{\Omega})$,*

$$\|u\|_2^p \le D\|(\Delta - \lambda)u\|^p$$

if $Bu(x) = 0$ and $\partial\Omega$, for all $p \ge 1$. The constant D depends on λ and Ω, but is independent of u.

To convert (1) into an operator equation, we choose $\lambda>0$ large enough so that $\lambda>M$ and λ is not in the spectrum of (1). Then by Theorem 1, we can define an operator T as follows: for any $u \in C^r(\bar{\Omega} : I)$, $0<r<1$,

$$Tu = v$$

if v is the unique solution of the boundary value problem :

$$\begin{cases} (\Delta - \lambda)v(x) = -f(x,u(x)) - \lambda u(x), & x \in \Omega \\ Bv(x) = \varphi(x), & x \in \partial\Omega \end{cases}$$

(For the Dirichlet problem case, we need the additional condition : $\varphi(x) \in I$ for all $x \in \partial\Omega$)

With the aid of Theorem 2, we note that T is a continuous operator from $C^r(\bar{\Omega} : I)$ into $W_p^2(\bar{\Omega})$ for $p \ge 1$ (Here $W_p^2(\bar{\Omega})$ is the set of all functions in $L_p(\bar{\Omega})$ possesing generalized derivatives of the first 2 orders in $\bar{\Omega}$ that are p-integrable over $\bar{\Omega}$). Since $C^r(\bar{\Omega} : I)$ is dense in $C(\bar{\Omega} : I)$ and T is continuous on $C^r(\bar{\Omega} : I)$, we can extend T to $C(\bar{\Omega} : I)$ continuously. We again denote the extension by T.

On the other hand, for p≥1 sufficiently large, $W_p^2(\bar{\Omega})$ is continuously embedded in $C^{1,\alpha}(\bar{\Omega})$. Therefore, we view $T: C(\bar{\Omega}: I) \rightarrow C(\bar{\Omega})$ as a continuous operator. By Theorem 2 and Ascoli-Azela Theorem, we can show that T is compact. Consequently, if u is a fixed point of the operator equation

$$Tu = u,$$

then u is a classical solution of (1).

Theorem 3 (Maximum Principles).

(A) *Let u satisfy the differential inequality*

$$(\Delta - \lambda)u(x) \geq 0$$

in some neighborhood in Ω*, where* $\lambda \geq 0$*. If u attains a nonnegative maximum D at an interior point of the neighborhood, then* $u(x) = D$ *for all x of the neighborhood.*

(B) *Let* $u \in C^2(\bar{\Omega})$*. Then for any* $\lambda \in \geq 0$*, let*

$$(\Delta - \lambda)u(x) \geq 0$$

for all $x \in \Omega$*. If u attains its maximum value D at a boundary point* x_0 *of* Ω*, then any outward directional derivative of u at* x_0 *is positive unless* $u(x) = D$ *for all* $x \in \bar{\Omega}$.

Using Theorem 3, we can show that T is an increasing operator in $C(\bar{\Omega}: I)$, i.e., if $u(x) \leq v(x)$ for all $x \in \bar{\Omega}$, then $Tu(x) \leq Tv(x)$ for all $x \in \bar{\Omega}$.

Definitions : A function $w: \bar{\Omega} \rightarrow R$ is a quasi-supersolution (or quasi-subsolution) of (1) in $\bar{\Omega}$, if for any $x_0 \in \bar{\Omega}$, there exists a neighborhod N of x_0 and a finite number of functions $w_k \in C^2(N)$, k=1,2,…,p such that

$$w(x) = \min_{1\leq k\leq p} w_k(x), \qquad (\text{or } \max_{1\leq k\leq p} w_k(x))$$

for all $x \in N$, where p may depend on x_0, and

$$\Delta w_k(x) + f(x, w_k(x)) \leq 0 \qquad (\text{or } \geq 0)$$

for all $x \in N \cap \Omega$ and k=1,2,…,p. Furthermore, if $x_0 \in \partial\Omega$,

$$p(x_0)w_k(x_0) + q(x_0)\frac{dw_k(x_0)}{d\nu} \geq \varphi(x_0) \quad (\text{or } \leq \varphi(x_0))$$

for all *k*.

Remark : Throughout this paper any pair of quasi-supersolution and quasi-subsolution is contained in the bounded subset $C^r(\bar{\Omega}: I)$, for some r, $0<r<1$ of $C(\bar{\Omega})$.

The following fact is the most important ingredient of this paper

Theorem 4. *Let* $\bar{w}$ *and* $\hat{w}$ *be a quasi-subsolution and a quasi-supersolution of (1), respectively. Then*

$$\bar{w}(x) \leq T\bar{w}(x) \quad \text{and} \quad \hat{w}(x) \geq T\hat{w}(x)$$

for all $x \in \bar{\Omega}$.

Proof : To show $\hat{w}(x) \geq T\hat{w}(x)$ for all $x \in \bar{\Omega}$, suppose that there is a point $x_0 \in \bar{\Omega}$ such that $\hat{w}(x_0) < T\hat{w}(x_0)$. Let $a = T\hat{w}(\hat{x}) - \hat{w}(\hat{x})$ be the positive maximum of $T\hat{w} - \hat{w}$. Then there exists a neightborhood $U_{\hat{x}}$ such that $x \in U_{\hat{x}}$ and $T\hat{w}(x) \leq \hat{w}(x) + a$ for all $x \in U_{\hat{x}}$.

Case 1 $\hat{x} \in \Omega$

By the definition, there exists a neighborhood $N_{\hat{x}}$ such that

$$\hat{x} \in N_{\hat{x}} \subset U_{\hat{x}} \quad \text{and} \quad \hat{w}(x) = \min_{1\leq k\leq p} w_k(x)$$

for all $x \in N_{\hat{x}}$. Let $\hat{w}(\hat{x}) = w_k(\hat{x})$ for some k. Then $T\hat{w}(x) \leq w_k(x) + a$ for all $x \in N_{\hat{x}}$ and $T\hat{w}(\hat{x}) = w_k(\hat{x}) + a$. Since $T\hat{w}(x) - w_k(x) \leq T\hat{w}(x) - \hat{w}(x)$ for all $x \in N_{\hat{x}}$,

$$\begin{aligned} T\hat{w}(\hat{x}) - w_k(\hat{x}) &= T\hat{w}(\hat{x}) - \hat{w}(\hat{x}) \\ &\geq T\hat{w}(x) - \hat{w}(x) \end{aligned}$$

$$\geq T\widehat{w}(x) - w_k(x)$$

for all $x \in N_{\hat{x}}$. Hence $T\hat{w}-w_k$ has the positive maximum value a at $\hat{x}$ in the neighborhood $N_{\hat{x}}$. On the other hand, in $N_{\hat{x}}$, if $\lambda > M$

$$(\Delta - \lambda)(T\widehat{w} - w_k)(x)$$

$$=[-f(x,\widehat{w}(x)) - \lambda\widehat{w}(x)] - \Delta w_k(x) + \lambda w_k(x)$$

$$\geq - f(x,\widehat{w}(x)) - \lambda\widehat{w}(x) + f(x,w_k(x)) + \lambda w_k(x)$$

$$\geq 0.$$

By Theorem 3, (A), $T\hat{w}(x)-w_k(x)=a$ for all x in $N_{\hat{x}}$. Hence, for all $x \in N_{\hat{x}}$,

$$a = T\widehat{w}(x) - w_k(x) \leq T\widehat{w}(x) - \widehat{w}(x) \leq T\widehat{w}(\hat{x})$$
$$- w_k(\hat{x}) = a.$$

Therefore, $T\hat{w}(x) = \hat{w}(x)+a$ for all $x \in N_{\hat{x}}$. By the continuation of the method on the boundary $\partial N_{\hat{x}}$ of $N_{\hat{x}}$, we can conclude that $T\hat{w}(x) = \hat{w}(x)+a$ for all $x \in \bar{\Omega}$. And so, for any $x \in \Omega$,

$$\Delta\widehat{w}(x) = \Delta T\widehat{w}(x)$$
$$= -f(x,\widehat{w}(x)) - \lambda\widehat{w}(x) + \lambda T\widehat{w}(x)$$
$$\geq -f(x,\widehat{w}(x)) + \lambda a$$

Since $\Delta\hat{w}(x) \leq -f(x, \hat{w}(x))$ localy in Ω, so

$$-f(x,\widehat{w}(x)) = -f(x,\widehat{w}(x)) + \lambda a$$

locally in Ω. This implies that a=0. This leads to a contradiction. So we can conclude that there is no interior point of Ω such that $T\hat{w}-\hat{w}$ has a positive maximum value at that point.

Case 2: $\hat{x} \in \partial\Omega$

Then there exists a neighborhood $N_{\hat{x}}$ such that $N_{\hat{x}} \cap \Omega \neq 0$, $x \in N_{\hat{x}}$, and $\hat{w}(x) = \min w_k(x)$ for all $x \in N_{\hat{x}} \cap \Omega$. Let $\hat{w}(\hat{x}) = w_k(\hat{x})$ for some k. Then $T\hat{w}(x) \leq w_k(x) + a$ for all $x \in N_{\hat{x}} \cap \bar{\Omega}$.

Since $T\hat{w}(x) = w_k(\hat{x}) + a$,

$$\frac{dT\widehat{w}(\hat{x})}{d\nu} \geq \frac{dw_k(\hat{x})}{d\nu}$$

If $p(\hat{x}) > 0$, then

$$\varphi(\hat{x}) \geq p(\hat{x})[w_k(\hat{x}) + a] + q(\hat{x})\frac{dw_k(\hat{x})}{d\nu}$$
$$\geq \varphi(\hat{x}) + p(\hat{x})a.$$

This leads to a contradiction for $p(\hat{x})a > 0$.

Let $p(\hat{x}) = 0$. Then $q(\hat{x}) > 0$.

If $\frac{dT\widehat{w}(\hat{x})}{d\nu} > \frac{dw_k(\hat{x})}{d\nu}$, then $\varphi(\hat{x}) > q(\hat{x})\frac{dw_k(\hat{x})}{d\nu} \geq \varphi(\hat{x})$. This also leads to a contradiction.

Let $\frac{dT\widehat{w}(\hat{x})}{d\nu} = \frac{dw_k(\hat{x})}{d\nu}$. For all $x \in N_{\hat{x}} \cap \Omega$

$$(\Delta - \lambda)(T\widehat{w} - w_k - a)(x)$$
$$\geq - f(x,\widehat{w}(x)) + f(x,w_k(x)) + \lambda[w_k(x) - \widehat{w}(x)] + \lambda a$$
$$\geq 0.$$

Since $T\hat{w}-(w_k+a)$ has the zero maximum value of the boundary point $\hat{x}$ in $N_{\hat{x}} \cap \Omega$, by Theorem 3, (B) $T\hat{w}(x) = w_k(x)+a$ for all $x \in N_{\hat{x}} \cap \Omega$. This implies that $T\hat{w}-\hat{w}$ has the positive maximum value a at an interior point of Ω. By Case 1, this also leads to a contradiction. Therefore, $T\hat{w} \leq \hat{w}(x)$ for all $x \in \bar{\Omega}$.

Similarly, we can shw that $T\bar{w}(x) \leq \bar{w}(x)$ for all $x \in \bar{\Omega}$.

Remark : Theorem 4 is valid if we replace Δ by a uniformly elliptic operator

$$L = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}(x)D^{ij} + \sum_{i=1}^{n} A_i(x)D^i + A_0(x)$$

where the coefficients of L and B are smooth, because Theorem 4 works for this perator with a large value of λ not in the spectrum of (1).

The following theorem is known

Theorem 5 [Amann, Deimling]. *Let X be a Banach space: $S\subset X$ a retract and $T: S\to S$ compact; S_1, S_2 disjoint retracts of S : $E_j\subset S_j$ open in S for $j=1,2$. Suppose that $T(S_j)\subset S_j$ and $\mathrm{Fix}(T)\cap(S_j/E_j)=\phi$ for $j=1,2$. Then T has fixed points $u_j\in E_j$ and a third fixed point $u_0\in S/(S_1\cup S_2)$, where $Fix(T)=\{u\in S: Tu=u\}$.*

The following theorem is the main result of this paper.

Theorem 6. *Suppose that $\overline{w}_1, \overline{w}_2$ are quasi-subsolutions and $\hat{w}_1, \hat{w}_2$ are quasi-supersolutions of (1) such that*

$$\overline{w}_1(x) \le \hat{w}_2(x) \quad \text{and} \quad \overline{w}_j(x) \le \hat{w}_j(x)$$

for all $x\in\overline{\Omega}$ and $j=1,2$, and $\overline{w}_2(x_0)>\hat{w}_1(x_0)$ for some $x_0\in\overline{\Omega}$. If $\hat{w}_1$ and $\overline{w}_2$ are not solutions of (1), then (1) has at least three distinct solutions u_j $(j=0,1,2)$, such that

$$\overline{w}_j(x) \le u_j(x) \le \hat{w}_j(x)$$

for all $x\in\Omega$ and $j=1,2$, and especially

$$u_0 \in [\overline{w}_1, \hat{w}_2] \setminus [\overline{w}_1, \hat{w}_1] \cup [\overline{w}_2, \hat{w}_2].$$

Proof : Without loss of generality, let $\varphi(x)=0$ for all $x\in\partial\Omega$.

Case 1 The Dirichlet boundary condition, i. e., $Bu=u$. Let e be the unique solution of the boundary value problem

$$\begin{cases} (\Delta-\lambda)e(x) = -1, & x\in\Omega \\ e(x)=0, & x\in\partial\Omega. \end{cases}$$

Consider the Banach space $C_e(\overline{\Omega})$, [Aman], which is the set of all functions u in $C(\overline{\Omega})$ so that $-ce(x)\le u(x)\le ce(x)$ for some constant $c\ge 0$ and for all $x\in\overline{\Omega}$, with the norm

$$\|u\|_e = \inf\{c>0 : -ce(x)\le u(x)\le ce(x), x\in\overline{\Omega}\},$$

we note that $C_e(\Omega)$ is continuously embedded in $C(\overline{\Omega})$ and the operator T maps $C_e(\overline{\Omega} : I)$ into $C_e(\overline{\Omega})$ compactly.

Let

$$S = C_e(\overline{\Omega})\cap[\overline{w}_1, \hat{w}_2], \quad S_j = C_e(\overline{\Omega})\cap[\overline{w}_j, \hat{w}_j],$$

$$(j=1,2,).$$

Clearly, $S_j\subset S$ and nonempty. From Theorem 4, $T(S)\subset S$ and $T(S_j)\subset S_j$ for $j=1,2$.

We note that S, S_1, and S_2 are retracts and $S_1\cap S_2=\phi$.

Let

$$E_1 = S_1\cap\{u\in C_e(\overline{\Omega}) : u(x)<\hat{w}_1(x), x\in\Omega\}$$

and

$$E_2 = S_2\cap\{u\in C_e(\overline{\Omega}) : u(x)>\overline{w}_2(x), x\in\Omega\}.$$

We note that E_1 and E_2 are open in S with respect to the norm $\|u\|_e$.

To show that $\mathrm{Fix}(T)\cap(S_j/E_j)=\phi$, $j=1,2$, suppose that there is $u\in\mathrm{Fix}(T)(S_j/E_j)$ for some j. Then

$$u\in S_j\setminus E_j \quad \text{and} \quad Tu=u.$$

Let $j=1$. We note that u is a solution of (1). Since $u\in S_1\setminus E_1$, $\overline{w}_1(x)\le u(x)\le\hat{w}_1(x)$ for all $x\in\overline{\Omega}$ and there is $x_0\in\Omega$ such that $u(x_0)=\hat{w}(x_0)$.

On some neighborhood of x_0, if we choose $\beta>0$ sufficiently large, but independent on λ, then

$$(\Delta-\beta)(u(x)-\hat{w}_1(x))$$

$$\ge -f(x,u(x))+f(x,\hat{w}_1(x))+\beta[\hat{w}_1(x)-u(x)]$$

≥ 0.

Since $(u-\hat{w}_1)(x) \leq 0$ for all x in the neighborhood, by Maximum Principles, $u(x) = \hat{w}_1(x)$ on the neighborhood of x_0. By the continuation of this method on the boundary of the neighborhood, we can conclude that $\hat{w}_1(x) = u(x)$ for all $x \in \bar{\Omega}$. This implies that $\hat{w}_1$ is a solution of (1). This leads to a contradiction because $\hat{w}_1$ is not a solution of (1). Hence $\mathrm{Fix}(T) \cap (S_1 \setminus E_1) = \phi$.

Similarly, we can show that $\mathrm{Fix}(T) \cap (S_2 \setminus E_2) = \phi$. Therefore T satisfies all conditions of Theorem 5. So T has at least three fixed points u_1, u_2, u_3 such that $u_j \in [w_j, w_j]$, $j=1,2$, and $u_3 \in [\bar{w}_1, \hat{w}_2]/[\bar{w}_1, \hat{w}_1] \cup [\bar{w}_2, \hat{w}_2]$.

Case 2 Neumann or Robin boundary conditions with $q(x) > 0$ for all $x \in \partial\Omega$.

In this case, we claim that for any $u \in C(\bar{\Omega} : I)$, (i) if $\bar{w}_2(x) \leq u(x)$ and $Tu(x) = u(x)$ for all $x \in \bar{\Omega}$, then $\bar{w}_2(x) < u(x)$ for all $x \in \bar{\Omega}$, and (ii) if $u(x) \leq \hat{w}_1(x)$ and $Tu(x) = u(x)$ for all $x \in \bar{\Omega}$, then $u(x) < \hat{w}_1(x)$ for all $x \in \bar{\Omega}$.

To prove the first claim, we suppose that there is $u \in C(\bar{\Omega} : I)$ such that $\bar{w}_2(x) \leq u(x)$ and $Tu(x) = u(x)$ for all $x \in \bar{\Omega}$ and $\bar{w}_2(x_0)$ for some $x_0 \in \bar{\Omega}$. If $x_0 \in \Omega$, then on some neighborhood of x_0,

$$(\Delta - \beta)(\overline{w}_2 - u)(x)$$
$$\geq - f(x,\overline{w}_2(x)) + f(x,u(x)) + \beta[u(x) - \overline{w}_2(x)].$$

So if we choose $\beta > 0$ sufficiently large, but independent on λ, then $(\Delta - \beta)(\bar{w}_2 - u)(x) \geq 0$ on the neighborhood of x_0. Since $(\bar{w}_2 - u)(x) \leq 0$ on the neighborhood, by Theorem 3, (A), $\bar{w}_2(x) = u(x)$ for all x of the neighborhood of x_0. By the continuation of this method on the boundary of the neighborhood, we can conclude that $\bar{w}_2(x) = u(x)$ for all $x \in \bar{\Omega}$. This leads to a contradiction, because $\bar{w}_2$ is not a solution of (1). Therefore $x_0 \in \partial\Omega$. Let $d\bar{w}_2(x_0)/d\nu > du(x_0)/d\nu$. Then $\varphi(x_0) < p(x_0)u(x_0) + q(x_0)\frac{d\bar{w}_2(x_0)}{d\nu} \leq \varphi(x_0)$: contradiction. Let $d\bar{w}_2(x_0)/d\nu = du(x_0)/d\nu$. Since on some neighborhood of x_0, $(\Delta - \beta)(\bar{w}_2 - u)(x) \geq 0$, by Theorem 3, (B), $\bar{w}_2(x) = u(x)$ on the neighborhood of x_0. Hence we can choose an interior point $\bar{x}$ of Ω such that $\bar{w}_2(\bar{x}) = u(\bar{x})$. As before, we can conclude that $\bar{w}_2(x) = u(x)$ for all $x \in \bar{\Omega}$. This leads to a contradiction. Therefore, $\bar{w}_2(x) < u(x)$ for $x \in \bar{\Omega}$.

The proof of the second claim is quit similar. Hence,

$$\mathrm{Fix}(T) \cap ([\overline{w}_1, \hat{w}_1] \setminus [\overline{w}_1, \hat{w}_1)) = \emptyset$$

and

$$\mathrm{Fix}(T) \cap ([\overline{w}_2, \hat{w}_2] \setminus (\overline{w}_2, \hat{w}_2]) = \emptyset.$$

By Theorem 4, T has at least three fixed points u_1, u_2, u_3 such that $u_j \in [\bar{w}_j, \hat{w}_j]$, $j=1,2$ and $u_3 \in [\bar{w}_1, \hat{w}_2]/[\bar{w}_1, \hat{w}_1] \cup [\bar{w}_2, \hat{w}_2]$.

3. Applications

Consider a class of singularly perturbed problems of this type :

$$\begin{cases} \varepsilon^2 \Delta u(x) + f(x,u(x)) = 0, & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \qquad (3)$$

where Ω is a smooth bounded open domain in R^n, $n \geq 1$, and the real-valued function $f : \bar{\Omega} \times I \to R^1$ is reguired to be C^1. More assumptions on f:

(f1) There exist $N (N \geq 2)$ functions $g_i : \bar{\Omega} \to I (i = 1, 2, \cdots, N)$, which belong to $C^2(\bar{\Omega})$ and $0 < g_1(x) < g_2(x) < \cdots < g_N(x)$ and $f(x, g_i(x)) = 0$ for all $x \in \bar{\Omega}$ and all $i = 1, 2, \cdots, N$

(f2) There exists a positive constant K such that $f_u(x, g_i(x)) \le -K^2$ for all $x \in \bar{\Omega}$ and all $i=1, 2, \cdots N$.

(f3) For $i=1$

$$\int_{\gamma(s)}^{g_1(s)} f(s,u)\,du > 0$$

for all $s \in \partial\Omega$ and $\gamma(s) \in [0, g_1(s))$. *For* $i \ge 2$, there exists an n-dimensional open subdomain Ω_i of Ω such that $\partial\Omega_i \in C^{2,\alpha}$ and $\int_{\gamma(s)}^{g_i(s)} f(s,u)\,du > 0$ for all $s \in \partial\Omega_i$ and $r(s) \in [0, g_i(s))$.

The main goal of this applications is to investigate how changes of sign of the function f give rise to the existence of several ***ordered positive solutions*** of (3) as well as to the formation of boundary layer for small values of the parameter $\varepsilon > 0$.

(3) has been discussed previously by Brown and Budin and Hess. Their proofs do not yield a complete ordering of solutions. More recently, de Figueiredo proved the existence of several ordered positive solutions under hypothes (f1), (f2), a much stronger condition than (f3), a certain symmetric property of Ω, and also he assumed f is independent of x. In that paper [De Figueiredo], he said that a complete ordering of positive solutions for general domain Ω in R^n, $n \ge 2$ is open.

Constructing N-pairs of quasi-subsolutions and quasi-supersolutions of (3) and using the main result of this paper, the complete ordering of positive solutions of (3) is obtained. In order to construct a quasi-subsolution and a quasi-supersolution of (3), we use a coordinate transformation near the boundary $\partial\Omega$. If $x \in \Omega$, we denote by $t(x)$ the distance from x to $\partial\Omega$ and by $s(x)$ the point of $\partial\Omega$ which is closest to x. Of course $s(x)$ might not be uniquely defined, but will be if x is close enough to $\partial\Omega$. For sufficiently small neighborhood of $\partial\Omega$, we have

$$\varepsilon^2 \Delta_x = \frac{\partial^2}{\partial \tau^2} + O(\varepsilon), \qquad t = \varepsilon\tau$$

This procedure is given in [Berger,Franenkel].

Now consider the following boundary value problem :

$$\begin{cases} \dfrac{\partial^2 u}{\partial \tau^2} = F(s, u(s,\tau)) \\ u(s,0) = \xi(s), \qquad u(s,\infty) = 0, \end{cases}$$

where $s \in \partial\Omega$ is a parameter and $\tau \in (0, \infty)$ and $F(s, \cdot)$ is a real-valued function defined on $[0, \infty)$.

The following fact is well known

Lemma. [Devillier, Fife]. *Let $\xi(s)$ and $F(s, u)$ be infinitely differential for $s \in \partial\Omega$, and $u \in (-\infty, +\infty)$, all derivatives being uniformly continuous. For all $s \in \partial\Omega$, assume that*

$$F(s,0) = 0, \quad F_u(s,0) > 0, \quad \int_0^{\omega} F(s,u)\,du > 0$$

for all $\in (0, \xi(s)]$ or $\omega \in [\xi(s), 0)$. Then there is a unique monotone solution $v(s,\tau)$ of the above boundary value problem and it is infinitely differentiable in s and u. Moreover each of the derivatives of v decays exponentially as $\tau \to \infty$, uniformly in s, in the sense that if D is any C^∞-linear differential operator in a and τ, then there exist positive constants C and β, possibly depending on D, such that $|Dv(s, \tau)| \le c \exp(-\beta\tau)$.

$$\varepsilon^2 \Delta_x v(s,\tau) = \frac{\partial^2 v}{\partial \tau^2} + O(\varepsilon)$$

as $\varepsilon \to 0$, uniformly on $s \in \partial\Omega$, if $v(s, \tau)$ is the unique monotone solution of the above boundary value problem.

Using Lemma, we have the following theorem which yields the existence of a pair of subsolution and supersolution of (3). [Kelley, Ko).

Theorem 7. *Let* $\partial\Omega \in C^{2,\alpha}$ *and* $f \in C^1(\overline{\Omega}\times I)$. *Suppose that hhere is a function* $g \in C^2(\overline{\Omega})$ *which satisfies the following conditions;*

(i) $g(x) > 0$ for all $x \in \partial\Omega$.

(ii) $f(x, g(x)) = 0$ for all $x \in \overline{\Omega}$

(iii) $f_u(x, g(x)) \le -K^2 < 0$ for all $x \in \overline{\Omega}$

(iv) $\int_{\gamma(s)}^{g(s)} f(s,u)\,du > 0$ for all $s \in \partial\Omega$ and $\gamma(s) \in [0, g(s))$.

Then there is a small positive number ε_0 *such that for any* ε, $0 < \varepsilon \le \varepsilon_0$, *there are a* C-*subsolution and a* C-*supersolution of* (3).

Proof : We note that there is a positive constant γ_1 such that $f_u(x, g(x) \pm \gamma) \le -K^2$ for all $0 \le \gamma \le \gamma_1$ and for all $x \in \overline{\Omega}$. Let $\rho > 0$ be a sufficiently small number, and let $\ell(\rho, s) = f(s, g(s) - \rho)$ for all $s \in \partial\Omega$. Then $\ell(\rho, s) > 0$ for all $s \in \partial\Omega$. Then there is a positive number $\gamma(\rho)$ such that $\ell(\rho, s) \ge \gamma(\rho)$ for all $s \in \partial\Omega$.

To show the existence of a subsolution and a supersolution of (3), let $F(s,u) = f(s, g(s) - \rho - u) - (\rho, s)$. The $F(s,0) = 0$ and $F_u(s,0) \ge K^2 > 0$ for all $s \in \partial\Omega$. Since $\ell(\rho, s) \to 0$ as $\rho \to 0$ for all $s \in \partial\Omega$, we can say that, for all $\rho > 0$ sufficiently small,

$$\int_0^w F(s,u)\,du = \int_{g(s)-\rho-w}^{g(s)-\rho} f(s,u)\,du - \ell(\rho,s)w$$

is positive for all $w \in (0, g(s) - \rho]$ and for all $s \in \partial\Omega$. Then, by Lemma, there is a unique monotone solution $v(s, \tau; \varepsilon)$ of the problem

$$\begin{cases} \dfrac{\partial^2 v}{\partial \tau^2} = f(s, g(s) - \rho - v) - \ell(\rho, s) \\ v(s, 0; \varepsilon) = g(s) - \rho \end{cases}$$

such that the first derivatives and the second derivatives and the second derivatives of v in s and τ decays exponentially as $\tau \to \infty$.

Let $\overline{O}_\rho = \{(s,t) : 0 \le t \le \rho, s \in \partial\Omega\}$ and $\rho \le t_*$, where t_* is independent of ε and is so chosen that the normals through distinct points of $\partial\Omega$ do not intersect on $\overline{O}_\rho$. We take a smooth function $\sigma(t) \in C^\infty((0,\infty))$ such that $\sigma(t) \equiv 1$ for $0 \le t \le \frac{\rho}{2}$ and $\sigma(t) \equiv 0$ for $t \ge \rho$, and $0 \le \sigma(t) \le 1$ for all t, and we define $\overline{v}(s, \tau; \varepsilon) = v(s, \tau; \varepsilon)\sigma(t)$ for $x \in \overline{O}_\rho$, where $t = \varepsilon\tau$.

Let $w(s, \tau; \varepsilon) = g(s, \tau) - \rho - \overline{v}(s, \tau; \varepsilon)$ for $x \in O_\rho$. Then for t, $0 \le t \le \rho$

$$\begin{aligned} \varepsilon^2 \Delta_x \overline{w}(s,\tau;\varepsilon) &= -\frac{\partial^2}{\partial \tau^2}\overline{v}(s,\tau;\varepsilon) + O(\varepsilon) \\ &= -[f(s, g(s) - \rho - v(s,\tau;\varepsilon)) \\ &\quad - \ell(\rho,s)]\sigma(t) + O(\varepsilon). \end{aligned}$$

Since we can assume that $f(s, g(s) - \rho - v(s,\tau;\varepsilon)) - \ell(\rho, s) > 0$ for all $\frac{\rho}{2} \le t \le \rho$ and for all $s \in \partial\Omega$,

$$\varepsilon^2 \Delta_x \overline{w}(s,\tau;\varepsilon) \ge -f(s, g(s) - \rho - v(s,\tau;\varepsilon)) + \ell(\rho,s) + O(\varepsilon)$$

for all t, $0 \le t \le \rho$, because $\sigma(t) \equiv 1$ for $0 \le t \le \frac{\rho}{2}$. Now

$$f(s,\tau,\overline{w}(s,\tau;\varepsilon)) - f(s,o,g(s) - \rho - v(s,\tau;\varepsilon))$$

$$= \nabla_x f(x^*(s), \overline{w}(s,\tau;\varepsilon)) \cdot \frac{dx}{dt}(t^*)t + f_u(s,u^*)$$

$$[1 - \sigma(t)]v(s,\tau;\varepsilon),$$

wher $x^*(s)$ is on the line segment passing through the point $x(s,\tau)$ and s on O_ρ, and $0 \le t < t^*$, and $\overline{w}(s,\tau;\varepsilon) < u^* < g(s) - \rho - v(s,\tau;\varepsilon)$. Sinve $v(s,\tau;\varepsilon)$ decays exponentially for $\frac{\rho}{2} \le t \le \rho$ as $\varepsilon \to 0$,

$$f(s,\tau,\overline{w}(s,\tau;\varepsilon)) - f(s,0,g(s) - \rho - v(s,\tau;\varepsilon))$$

$$= O(t).$$

So we have

$$\varepsilon^2\Delta_s\overline{w}(s,\tau;\varepsilon) \geq -f(s,\tau,\overline{w}(s,\tau;\varepsilon)) + \ell(\rho,s) + O(t)$$

Since $\ell(\rho,s) > \gamma(\rho) > 0$ for all $s \in \partial\Omega$, there is a positive number t_0 such that for all t, $0 \leq t \leq t_0$,

$$\varepsilon^2\Delta_s\overline{w}(s,\tau;\varepsilon) + f(s,\tau,\overline{w}(s,\tau;\varepsilon)) \geq 0.$$

For t, $t_0 < t \leq \rho$, we note that $\overline{w}(s,\tau;\varepsilon)$ converges to $g(s,\tau) - \rho$ as $\varepsilon \to 0$, uniformly on $s \in \partial\Omega$. So for sufficiently small $\varepsilon > 0$,

$$\varepsilon^2\Delta_s\overline{w}(s,\tau;\varepsilon) + f(s,\tau,\overline{w}(s,\tau;\varepsilon)) \geq 0. \qquad (*)$$

For $x \in \Omega \setminus \overline{O}_\rho$, $\overline{w}(x;\varepsilon) = g(x) - \rho$. Then

$$\varepsilon^2\Delta\overline{w}(x;\varepsilon) + f(x,\overline{w}(x;\varepsilon)) = O(\varepsilon^2) + f(x,g(x) - \rho) \qquad (**)$$

for all $\varepsilon > 0$ sufficiently small. Combining $(*)$ and $(**)$,

$$\varepsilon^2\Delta\overline{w}(x;\varepsilon) + f(x,\overline{w}(x;\varepsilon)) \geq 0$$

for all $x \in \overline{\Omega}$ and for all $\varepsilon > 0$ sufficiently small. Since $\overline{w}(s,0;\varepsilon) = 0$, $\overline{w}(x;\varepsilon)$ is a subsolution of (3).

To construct a supersolution, we choose a constant $\beta_1 > 0$ such that $f_u(x, g(x) + \beta) \leq -K^2$ for all β with $0 \leq \beta \leq \beta_1$. Let $\hat{w}(x;\varepsilon) = g(x) + \beta_1$. Then

$$\varepsilon^2\Delta\hat{w}(x;\varepsilon) + f(x,\hat{w}(x:\varepsilon))$$
$$= O(\varepsilon^2) + f(x,g(x) + \beta_1) \leq 0$$

for all $\varepsilon > 0$ sufficiently small and for all $x \in \overline{\Omega}$. Since $\hat{w}(x;\varepsilon) \geq g(x)$, $x \in \partial\Omega$, $\hat{w}$ is a supersolution of (3).

Remark: Let $\partial\Omega \in C^{2,\alpha}$, $f \in C^\alpha(\overline{\Omega} \times I)$, $0 < \alpha < 1$. If we replace the hypothesis (iii) in Theorem 7 by the assumption that there is a positive number γ_1 such that

$$f(x,g(x) - \gamma) > 0 \quad \text{and} \quad f(x,g(x) + \gamma) < 0$$

for all $x \in \overline{\Omega}$ and for all γ with $0 < \gamma \leq \gamma_1$, then the claim is also true.

The following theorem is the complete answer to the question of the existence of $2N-1$ ordered positive solutions of (3) for a smooth bounded open domain Ω of R^n, $n \geq 1$.

Theorem 8. *Let $\partial\Omega \in C^{2,\alpha}$, $f \in C^1(\overline{\Omega} \times I)$ and f satisfy (f1), (f2) and (f3). Then there exists $\varepsilon_0 > 0$ such that for all all ε with $0 < \varepsilon \leq \varepsilon_0$, the (3) $2N-1$ ordered positive solutions $u_i(x;\varepsilon)$, $u_{i+\frac{1}{2}}(x;\varepsilon)$, $u_{i+1}(x;\varepsilon)$, ($i = 1, \cdots, N-1$) such that*

(i) $u_i(x;\varepsilon) \leq u_{i+\frac{1}{2}}(x;\varepsilon) \leq u_{i+1}(x;\varepsilon)$ for all $x \in \overline{\Omega}$,

(ii) for any integer j, ($j = 1, \cdots, N$), $u_j(x;\varepsilon)$ converges to $g_j(x)$ as $\varepsilon \to 0$, uniformly on every compact subset of Ω_j (Here $\Omega_1 = \Omega$).

Proof: Let Ω_i, $i = 1, 2, \cdots, N$ be the subdomain in hypothese (f3) and let $\Omega_1 = \Omega$. For any i, by Theorem 7, there exists $\varepsilon_i > 0$ such that for any ε with $0 < \varepsilon \leq \varepsilon_i$, there is a pair of functions $\overline{w}_i(x;\varepsilon)$ and $\hat{w}_i(x;\varepsilon)$ such that

$$\overline{w}_i(x;\varepsilon) \leq \hat{w}_i(x;\varepsilon)$$

for all $x \in \Omega_i$ and

$$\begin{cases} \varepsilon^2\Delta\overline{w}_i(x;\varepsilon) + f(x,\overline{w}_i(x;\varepsilon) \geq 0, & x \in \Omega_i \\ \overline{w}_i(x;\varepsilon) = 0, & x \in \partial\Omega_i \end{cases}$$

and

$$\begin{cases} \varepsilon^2\Delta\hat{w}_i(x;\varepsilon) + f(x,\hat{w}_i(x;\varepsilon) \leq 0, & x \in \Omega_i \\ \hat{w}_i(x;\varepsilon) \geq g_i(x), & x \in \partial\Omega_i \end{cases}$$

Let $\varepsilon_0 = \min\{\varepsilon_i \mid 1 \leq i \leq N\}$, and for any ε, $0 < \varepsilon \leq \varepsilon_0$, let

$$\bar{u}_1(x;\varepsilon) = \overline{w}_1(x;\varepsilon), \qquad x \in \overline{\Omega}$$

$$\bar{u}_2(x;\varepsilon) = \begin{cases} \max\{\bar{u}_1(x;\varepsilon), \overline{w}_2(x;\varepsilon)\}, & x \in \overline{\Omega}_2 \\ \bar{u}_1(x;\varepsilon), & \text{otherwise} \end{cases}$$

$$\bar{u}_3(x;\varepsilon) = \begin{cases} \max\{\bar{u}_2(x;\varepsilon), \overline{w}_3(x;\varepsilon)\}, & x \in \overline{\Omega}_3 \\ \bar{u}_2(x;\varepsilon), & \text{otherwise} \end{cases}$$

$$\tilde{u}_n(x;\varepsilon) = \begin{cases} \max\{\tilde{u}_{n-1}(x;\varepsilon), \overline{w}_n(x;\varepsilon)\}, & x \in \overline{\Omega}_n \\ \tilde{u}_{n-1}(x;\varepsilon), & \text{otherwise} \end{cases}$$

Then, $\bar{u}_i(x;\varepsilon) \le w_i(x;\varepsilon)$ for all $x \in \bar{\Omega}$, and it is a pair of quasi-subsolution and supersolution of (3). By the construction $\bar{u}_i(x;\varepsilon) \le \hat{w}_{i+1}(x;\varepsilon)$ for all $x \in \bar{\Omega}$ and all $i=1,2,\cdots,N-1$, as long as ε_0 is sufficiently small. Clearly, $\hat{w}_i(x;\varepsilon)$ and $\bar{u}_{i+1}(x;\varepsilon)$ are not solutions of (3).

By theorem 6, there exist at least $2N-1$ solutions $u_i(x;\varepsilon)$, $u_{i+\frac{1}{2}}(x;\varepsilon)$, $u_{i+1}(x;\varepsilon)$ of (3) for all $i=1,2,\cdots N-1$ such that

$$u_i(x;\varepsilon) \le u_{i+\frac{1}{2}}(x;\varepsilon) \le u_{i+1}(x;\varepsilon)$$

$$\bar{u}_i(x;\varepsilon) \le u_i(x;\varepsilon) \le \hat{w}_i(x;\varepsilon).$$

for all $x \in \bar{\Omega}$ and $u_{i+\frac{1}{2}} \in [\bar{u}_i, \hat{w}_{i+1}] \setminus [\bar{u}_i, \hat{w}_i] \cup [\bar{u}_{i+1}, \hat{w}_{i+1}]$. We note that u_i is the minimal solution and u_{i+1} is the maximal solution of (3) in the order interval $[\bar{u}_i, \hat{w}_{i+1}]$. By the construction of $\bar{w}_i$, $\bar{u}_i$ and $\hat{w}_i$, $u_i(x;\varepsilon)$ converges to $g_i(x)$ as $\varepsilon \to 0$, uniformly on every compact subset of Ω_i for $i=1,2,\cdots,N$.

Remark: As before the remark of Theorem 7, if we replace the hypothesis (f2) in Theorem 8 by the assumption that there is a positive number γ_1 such that

$$f(x, g_i(x) - \gamma) > 0 \quad \text{and} \quad f(x, g_i(x) + \gamma) < 0$$

for all $x \in \bar{\Omega}_i$ and for all γ with $0 < \gamma \le \gamma_1$, then the claim of Theorem 8 is also true.

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〈國文抄錄〉

반선형인 타원형 문제에서 중간해의 존재성

반선형인 타원형 경계치 문제 $\Delta u+f(x, u(x))=0$, $x\in\Omega$, $Bu(x)=\phi(x)$, $x\in\partial\Omega$에서 다수의 해들이 존재할 수 있다는 사실을 증명하고, 그 결과의 응용으로써 특이섭동반선형인 경계치 문제에서 순서적으로 양의 해들의 존재성을 증명하며, 그 해들 중에는 경계현상을 갖는것, 비일양 내부현상을 갖는 해들이 존재함을 증명한다.