Restricted Generalized Inverse의 제한된 최소화 문제에의 응용에 관한 연구

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A Restricted Generalized Inverse and its Application to Constrained Minimization Problem

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본 논문에서는 restricted generalized inverse를 소개하고 이것에 대한 응용으로 제한된 선형 연산자의 minimization 문제에 관한 유일한 해의 존재 조건과 표현 양상에 대해서 알아보았다.

1. Introduction

Suppose that H_1 and H_2 are Hilbert spaces over the same scalar fields.

We consider a linear equation of the type

$$Tx = b \text{ where } b \in H_2,\ T \in L(H_1, H_2) \qquad (1)$$

If T has an inverse then equation (1) always has the unique solution $x = T^{-1}b$.

Moreover. (1) may have more than one solution (If $N(T) \neq \{0\}$)

or may have no solution at all(If $b \notin R(T)$).

Even if the equation (1) has no solution in the traditional sense it is still possible to assign what is in a sense a best possible solution to problem. It seems reasonable the consider

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a generalized solution of (1).

In this paper, first, we introduce the concept of generalized inverse and restricted generalized inverse.

Minamide and Nakamura introduced recently the concept of the restricted generalized inverse which possesses a constrained best approximation property and which has applications to certain constrained minimization problems.

In the second part, we will investigate the solution of following minimization problem ;

Let $A: X \to Y$, $L: X \to Z$ be linear operators where X, Y, Z are Hilbert spaces and L has a closed range

Find $w \in G_y = \{u \in x : \|Au-y\| = \inf \|Ax-y\|\}$ such that

$$\|Lw\| = \inf\{\|Lu\| : u \in G_y\} \text{ where } x \in X,\ y \in Y.$$

2. Preliminaries

1) Generalized inverses of linear operators on Hilber spaces.

Let X and Y be Hilbert spaces. Let $T: X \to Y$ be a linear operator. We denote the range of by R(T), the null space of T by N(T) and the adjoint of T by T^*. For any subspace S of a Hilbert space H we denotes the orthogonal complement of S by S^2 and the closure of S by $\bar{S}$.

Now, consider an operator equation of the first kind:

$$Tx = y, \text{ where } x \in X,\ y \in Y. \quad (1.1)$$

Definition 1.1 : For a given $y \in Y$, an element $u \in X$ is called a least squares solution of (1.1) if and only if $\|Tu-y\| \leqslant \|Tx-y\|$ for all $x \in X$.

Definition 1.2 : An element $\bar{u}$ is called a least squares solution of minimal norm of(1.1) if and only if $\bar{u}$ is a least squares solution of(1.1) and $\|\bar{u}\| \leqslant \|u\|$ for all least squares solutions u of(1.1).

Definition 1.3 : The generalized inverse of T, denoted by T^+, defined by $T^+y = \bar{u}$. where $\bar{u}$ is the least squares solution of minimal norm of $Tx = y$.

If T is continuous, then N(T) is closed. The adjoint T^* is also bounded and the following relations are valid ;

$$\overline{R(T)}=N(T^*)^{\perp},\ R(T)^{\perp}=N(T^*)$$
$$\overline{R(T^*)}=N(T)^{\perp},\ R(T^*)^{\perp}=N(T)$$

(See Groetsch [1, p14]).

Let $T:X\rightarrow Y$ be a bounded linear operator and let P be the projection of Y onto $\overline{R(T)}$, then we can easily check that the following conditions on $u\in X$ are equivalent ;

(1) $Tu=Py$

(2) $\|Tu-y\|\leqslant\|Tx-y\|$ for all $x\in X$

(3) $T^*Tu=T^*y$ where $y\in Y$

(See Groetch [1, p114])

In other word, $u\in X$ is a least squares solution of the equation $Tx=y$ if and only if u satisfies the equivalent conditions (1)-(3).

We can easily prove that the following statements are true.

(1) If u is a least squares solution of $Tx=y$ then $u+x$ is also least squares solution of $Tx=y$, where $x\in N(T)$.

(2) If u is a least squares solution of $Tx=y$ and $u=x_1+x_2$ where $x_1\in N(T)^{\perp}$, $x_2\in N(T)$, then $x_1=T^+y$.

(3) For all least squares solution u, $u=T^+y+x_1$, where $x_1\in N(T)$.

2) The restricted generalized inverses of bounded linear operators with closed ranges.

Let X, Y and Z be Hilbert spaces over the same scalar fields, and let $T: X\rightarrow Y$, $S: X\rightarrow Z$ be bounded linear operators with closed ranges.

Consider the Product transformation (T, S) on X into $Y\times Z$ defined by $(T, S)u=(Tu, Su)$, where $Y\times Z$ is the product Hilbert space equipped with the usual inner product.

Assume that the transformation (T, S) has a closed range in $Y\times Z$.

Since S is continuous, the null space N(S) is a closed subspace of X.

Denote the restriction of T onto N(S) by T_S.

Since T is a bounded linear operator with closed range.

$T_S: N(S)\rightarrow Y$ has also a closed range in Y.

So, for all $y\in Y$, there exists a unique T_S^+y.

3. Constrained minimization problems.

Let X, Y, Z be three Hilbert spaces and $A: X \to Y$, $L: X \to Z$ are bounded linear operators and $R(L)$ is closed.

We consider the following minimization problem;

Let $G_y = \{u \in X : \| Au-y \| = \inf \| Ax-y \|, x \in X\}$ where $y \in Y$. Find $w \in G_y$ such that $\| Lw \| = \inf \{\| Lu \| : u \in G_y\}$.

Lemma 2.1 G_y is a non-empty closed convex subset of X and $u \in G_y$ if and only if $u = A^+y + (I-A^+A)v$ for some $v \in X$.

proof) For all $u \in G_y$, u can be represented by $u = A^+y + u_1$ where $u_1 \in N(T)$

i. e. $G_y = \{A^+ y + u_1 : u_1 \in N(T)\}$

Since $N(T)$ is a closed convex subset of X, G_y is also a non-empty closed convex subset of X.

($\Rightarrow$) Suppose that $u = A^+y + u_1$ where $u_1 \in N(T)$.

Then $u_1 = (I-A^+A)u_1$. Thus let $v = u_1$ then $u = A^+y + (I-A^+A)v$

($\Leftarrow$) Let $u = u_1 + u_2$ where $u_1 \in N(T)^\perp$, $u_2 \in N(T)$.

Then $(I-A^+A)u = u_1 + u_2 - u_1 = u_2 \in N(T)$.

Thus $A^+y + u_2 \in G_y$.

Proposition 2.2. $\| Lw \| = \{\| Lu \| : u \in G_y\}$ if and only if $A^*Aw = A^*y$ and $L^*Lw \in N(A)^\perp$

Proof) Let $w \in G_y$ which satisfies $\| Lw \| = \{\| Lu \| : u \in G_y\}$. Then $A^* Aw = A^*y$ is obvious.

For all $u \in G_y$, u can be represented by $u = A^+y + u_1$, where $u_1 \in N(A)$, and denotes $w = A^+ y + w_1$, where $w_1 \in N(A)$.

Since $\| Lw \| = \inf \{\| Lu \| : u \in G_y\}$, $\| L(A^+y + w_1) \| < L(A^+{}_y + u_1) \|$ for all $u_1 \in N(A)$

i. e. $\| L(A^+y) + L(w_1) \| \leqslant \| L(A^+y) + L(u_1) \|$ for all $u_1 \in N(A)$ — (1)

Now consider the restriction of L onto $N(A)$, denoted by L_A

Then (1) induces that $\| L_A(w_1) + L(A^+y) \| \leqslant \| L_A(u_1) + L(A^+y) \|$ for all $u_1 \in N(A)$.

Since L has a closed range, L_A has also a closed range.

It shows that $w_1 = L_A^+(-L(A^+y)$. Consequently

$L_A(w_1) + L(A^+y) = L(w_1 + A^+y) \in R(L_A)^\perp$.

Thus for all $u_1 \in N(A)$, $L_A(u_1) \in R(L_A)$ and

$(L_A(u_1), L_A(w_1) + L(A^+y)) = (L_A(u_1), L(A^+y + w_1))$

$= (L(u_1), L(w_1 + A^+y)) = (u_1, L^*L(w_1 + A^+y)) = (u_1, L^*L(w)) = 0$.

Namely, $L^*L(w) \in N(A)^\perp$

Proposition 2.3. $w \in G_y$ which satisfies $\| Lw \| = \inf \{ \| Lu \| : u \in G_y\}$ is unique if and only if $N(A) \cap N(L) = \{0\}$

proof) ($\Rightarrow$) Suppose that $N(A) \cap N(L) \neq \{0\}$

Then there exists at least one $w_2 \in N(A) \cap N(L)$ which is not zero.

Thus $\| Aw-y \| = \| A(w+w_2)-y \| \leqslant \| Ax-y \|$ for all $x \in X$. Consequently $w+w_2$ also belongs to G_y and $\| L(w+w_2) \| = \| L(w) \| = \inf \{ \| Lu \| : u \in G_y\}$

($\Leftarrow$) Suppose that $N(A) \cap N(L) = \{0\}$

Then $N(L_A) = \{0\}$ and since $\| Lw \| = \inf\{ \| Lu \| : u \in G_y\}$ implies that

$\| L(A^+y + w_2) \| = \inf\{ \| L(A^+y + w_1) \| : w_1 \in N(A)\}$

$\| L(A^+y + w_2) \| = \| L(w_2) + L(A^+y) \| = \inf\{ \| L(w_1) + L(A^+y) \| : w_1 \in N(A)\}$

where $w = A^+y + w_2$

Consequently, we can represent w_2 as $L^+{}_A(-L(A^+y)) + p$, where $p \in N(L_A)$. But since $N(L_A) = \{0\}$, w_2 is unique.

Theorem 2.4. Let $G_y = \{u \in X : \| Au-y \| = \inf \| Ax-y \|, x \in X\}$ and $N(A) \cap N(L) = \{0\}$, Then there exists a unique $w \in G_y$ such that

$\| Lw \| = \inf\{ \| Lu \| : u \in G_y\}$ and $w = A^+y + L^+{}_A\{-L(A^+y)\}$

proof) Since $u \in G_y$ is denoted by $u = A^+y + u_1$, where $u_1 \in N(A)$.

Let $w = A^+y + w_1$ where $w_1 \in N(A)$.

Then $\| Lw \| = \| L(w_1) + L(A^+y) \| \leqslant \inf\{ \| Lu \| : u \in G_y\}$

$= \inf\{ \| L(u_1) + L(A^+y) \| : u_1 \in N(A)\}$

Since L has a closed range L^+y always exists for all $y \in Z$

Thus by Proposition 2.2, 2.3 the unique $w \in G_y$ which $\| Lw \| = \inf\{ \| Lu \| : u \in G_y\}$ exists.

Now, $\| L(w_1) + L(A^+y) \| \leqslant L(u_1) + L(A^+y)$ for all $u_1 \in N(A)$ implies that

$\| L_A(w_1)+L(A^+y) \| = \inf\{ \| L_A(u)+L(A^+y) \| : u_1 \in N(A)\}$

In result, u_1 is a least squares solution of the equation $L_A x+L(A^+y)=0$ and by uniqueness, u_1 itself is the least squares solution of minimal norm.

Thus $w_1 = L^+{}_A(-L(A^+y))$ and $w = A^+y + L^+{}_A(-L(A^+y))$.

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