제한된 최소화 문제에 있어서 STEEPEST DESCENT METHOD와 수치해석적인 접근 방법에 관한 연구

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THE STEEPEST DESCENT METHOD AND NUMERICAL APPROXIMATIONS FOR CONSTRA INED MINIMIZATION PROBLEMS

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초 록

본 논문에서는 제한된 최소화 문제의 해의 존재성과 유일성에 관한 조건들을 연구하고, 수치 해석적 접근방법으로 STEEPEST DESCENT METHOD를 사용하여 근사해를 구하는 방법을 얻었다.

I. Introduction

In author's Master thesis[1], the concept of generalized inverse and restricted generalized inverse were introduced. The concept of the restricted generalized inverse possesses a constrained best approximation property and has applications to certain constrained minimization problems.

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In this paper, solutions of constrained minimization problem are studied ; Among all least squares solutions of $Lx=z$, find an element w which minimizes $\| Ax-y \|$, where X, Y, Z are Hilbert spaces, $A: X\to Y$, $L: X\to Z$ are bounded linear operators and $x\in X$, $y\in Y$, $z\in Z$.

Especially, as one of the numerical method, the method of steepest descent is used to anlayze their approximastion.

2. The Restricted Generalized Inverses of Bounded Linear Operators with Closed Ranges

The restricted generalized inverse possesses the following constained best approximate solution property : let $y\in Y$ and $\bar{u}=T_S^+ y$. Then

(1) $S\bar{u}=O$.

(2) $\| T\bar{u}-y \| \leqslant \| Tu-y \|$ for all $u\in N(S)$.

(3) $\| \bar{u} \| \leqslant \| u \|$ for all $u\in N(s)$ such that $\| T\bar{u}-y \| = \| Tu-y \|$.

Proposition 1. Let $T: X\to Y$, $S: X\to Z$ be two bounded linear operators with closed ranges, and let $z\in R(S)$ and $y\in Y$. Then there exists a unique element $\bar{u}\in X$ satifying the following conditions.

(1) $S\bar{u}=z$.

(2) $\| T\bar{u}-y \| \leqslant \| Tu-y \|$ for all $u\in \{u: Su=z,\ u\in X\}$.

(3) $\| \bar{u} \| \leqslant \| u \|$ for all such u with $\| T\bar{u}-y \| = \| Tu-y \|$

and $\bar{u}=T_S^+ (y-TS^+z)+S^+z$, where T is the restriction of T onto N(S).

Proof) Since T and S have closed ranges, thus the exixtence is obvious. Now let

$$Wz=\{S^+z+x : x\in N(S)\}.$$

Then since $S\bar{u}=z$ thus $\bar{u}\in Wz$. Let $\bar{u}=S^+z+u_1$, where $u_1\in N(S)$. Then by condition(2),

$$\| T(S^+z+u_1)-y \| \leqslant \| T(S^+{}_z+x)-y \|$$

for all $x\in N(S)$ if and only if

$$\| T(u_1)-\{y-T(S^+z)\} \| \leqslant \| T(x)-\{y-T(S^+z)\} \|$$

for all $x\in N(S)$. T_s has closed range thus $y-T(S^+{}_z)\in D(T_s)$. Namely, u_1 can be represented by $T^+{}_s\{y-T(S^+z)\}+P$, where $P\in N(T)\cap N(S)$. But by condition(3), necessarily, $\bar{u}=S^+z+T^+{}_s\{y-T(S^+z)\}$.

3. Constrained Minimiztion Problems in Bounded Linear Operators with Arbitrary Ranges

Let X, Y, Z be three(real or complex) Hilbert spaces, and $A: X \to Y$, $L: X \to Z$ are bounded linear operators and R(L) is closed. We consider the following minimization problem :

Among all least squares solutions of $Lx=z$, find an element w which minimizes

$$\| Ax-y \| .$$

Proposition 2. Let $Wz=\{x \in X : x \text{ is a least aquares solution of } Lx=z, z \in Z\}$. Then $w \in Wz$ such that $\| Aw-y \| \leqslant \| Ax-y \|$ for all $x \in Wz$, where $y \in Y$ if and only if $A^*Aw-A^*y \in N(L)$.

Proof) Since every least squares solution of $Lx=z$ can be represented by L^+z+w, where $w \in N(L)$, $Wz=\{L^+z+w_1 : w_1 \in N(L)\}$.

Now, let $w \in Wz$ such that $\| Aw-y \| \leqslant \| Ax-y \|$ for all $x \in Wz$. Then

$$\| A(L^+z+w_1)-y \| \leqslant \| A(L^+z+x_1)-y \|$$

for all $x_1 \in N(L)$, where $w=L^+z+w_1$. It shows that

$$\| Aw_1-\{y-A(L^+z)\} \| < \| Ax_1-\{y-A(L^+z)\} \| \quad \cdots\cdots\cdots (1)$$

for all $x_1 \in N(L)$. Note that N(L) is a closed subspace of X.

Now, consider the restriction of A onto N(L), denoted by A_L. Since $Y=\overline{R(A_L)}+R(A_L)^\perp$, the above condition(1) is equivalent to $Aw_1-\{y-A(L^+z)\} \in R(A_L)^\perp$. Thus for all $x \in N(L)$

$$(Ax, Aw_1-\{y-A(L^+z)\})=0$$

if and only if

$$(x, A^*Aw_1-A^*\{y-A(L^+z)\})=0$$

for all $x \in N(L)$. Namely, $A^*Aw-A^*y \in N(L)^\perp$.

Proposition 3. There exists a unique $w \in Wz$ if and only if $N(A) \cap N(L)=\{0\}$.

Proof) ($\Leftarrow$). Suppose that $N(A) \cap N(L)=\{0\}$. Then since $N(A_L)=\{0\}$ thus there exists a unique $w_1 \in N(L)$ such that $\| Aw_1-\{y-\{y-A(L^+z)\}\} \| \leqslant \| Ax_1-\{y-A(L^+z)\} \|$ for all $x_1 \in N(L)$. It shows that there exists a unique $w=L^+z+w_1 \in Wz$ such that $\| Aw-y \| \leqslant \| Ax-y \|$ for all $x \in Wz$.

($\Rightarrow$). Suppose that $N(A) \cap N(L)=\{0\}$ then there exists at least one $w_2 \in N(A) \cap N(L)$ whch is not zero. Thus, $\| Aw-y \| = \| A(w+w_2)-y \| \leqslant \| Ax-y \|$ for all $x \in Wz$.

Consequently, w is not unique.

Proposition 4. Let $wz=\{x\epsilon X: x$ is a least squares solution of $Lx=z\}$and let A_L be the restriction of A on N(L). Suppose that $y-A(L^+z)\epsilon D(A_L^+)$ and $N(A)\cap N(L)=\{0\}$. Then there exists a unique $w\epsilon Wz$ such that $\|Aw-y\| \leqslant \|Ax-y\|$ for all $x\epsilon Wz$ and $w=A^+{}_L\{y-A(L^+z)\}+L^+z$.

Proof) Since $y-A(L^+z)\ \epsilon\ D(A^+{}_L)$, by Proposition 2 and Proposition 3 there exists a unique $w\epsilon Wz$ such that $\|Aw-y\| \leqslant \|Ax-y\|$ for all $x\epsilon Wz$.

Now, suppose that $w=w_1+L^+z\epsilon Wz$ such that $\|Aw-y\| \leqslant \|Ax-y\|$ for all $x\epsilon Wz$. Then

$$\|Aw_1-\{y-A(L^+z)\}\| \leqslant \|Ax_1-A(L^+z)\}\|$$

for all $x_1\epsilon N(L)$. Thus $w_1=A_L^+\{y-A(L^+z)\}$. Consequentoy, $w=A_L^+\{y-A(L^+z)\}+L^+z$.

Theorem 5. Let X,Y, Z, be Hilbert spaces and let $A: X\rightarrow Y$, $L: X\rightarrow Z$ be bounded linear operators, where R(L) is closed. Then the following conditions are equivalent ;

(1) There exists $w\epsilon Wz$ such that $\|Aw-y\| \leqslant \|Ax-y\|$ for all $x\epsilon Wz$, where $y\epsilon Y$.

(2) $A^*Aw-A^*y\epsilon N(L)^\perp$

(3) $y-A(L^+z)\epsilon D(A^+{}_L)$, where A_L is the restriction of A on N(L).

Proof) By Proposition 2 and 4, the proof is so easy.

Thoerem 6. Let X,Y and Z be Hilbert spaces, and $A: X\rightarrow Z$, $L: X\rightarrow Z$ be bounded linear operators, where R(L) is closed. Suppose that $N(L)\cap N(A)=\{0\}$ and R(A) is closed, then for all $y\epsilon Y$ and $z\epsilon Z$ there exists a unique $w\epsilon Wz$ such that

$$\|Aw-y\| \leqslant \|Ax-y\|$$

for all $x\epsilon Wz$, where $Wz=\{x\epsilon X: x$ is a least squares solution of $Lx=z\}$.

Proof) By assumption, since $R(A_L)$ is closed thus for all $y\epsilon Y$, $y-A(L^+z)\epsilon D(A^+{}_L)$ and by Proposition 4 there exists a unique $w\epsilon Wz$ such that $\|Aw-y\| \leqslant \|Ax-y\|$ for all $x\epsilon Wz$.

Theorem 7. Let $N(L)\cap N(A)=\{0\}$ and let $R(A_L)$ be closed and suppose that (x_n) is a sequence of approximations which converges to L^+z, then we can find an approximation $\bar{x}$ of w which $\|w-\bar{x}\| < \epsilon$ for arbitrary $\epsilon > 0$, where $w\epsilon Wz$ such that $\|Aw-y\|$ for all $x\epsilon Wz$.

Proof) By assumption, A_L has a closed range. It shows that $A^+{}_L$ is bounded. Since x_n converges to L^+z, for arbitrary $\varepsilon > 0$ we can take x_n such that

$$\| L^+z-x_n \| < \min \left(\frac{\varepsilon}{2\|A_L^+\| \|A\|}, \frac{\varepsilon}{2} \right)$$

Since $y-A(x_n) \in D(A^+{}_L)$, let $\bar{x} = A^+{}_L\{y-A(x_n)\}+x_n$, then

$$\begin{aligned} \| w-\bar{x} \| &\leqslant \| A^+{}_L\{A(L^+z-x_n)\} \| + \| L^+z-x_n \| \\ &\leqslant \| A^+{}_L \| \, \| A \| \; Lz+Lz-x \\ &< \epsilon . \end{aligned}$$

Namely,

$$\| w-\bar{x} \| < (\| A^+{}_L \| \, \| A \| +1) \| L^+z-x_n \| .$$

As an application, we consider an example by using of steepest descent method.

Proposition 8 The sequence (x_n) converges to L^+z, where $x_{n+1} = x_n - a_n r_n$, $r_n = L^*Lx-L^*z$, $a_n = \| r_n \|^2 / \| Lr_n \|^2$. Its speed of convergence is given by the inequality

$$\| x_n-L^2z \| \leqslant C \left(\frac{M-m}{M+m} \right)^n \qquad (n=0, 1, 2, \cdots : C= \qquad)$$

Proof) Since R(L) is closed, Lz exists for all $z \in Z$. This method is steepest descent method and the convergence is obvious. For the detail proof, see Kantovich[3, p. 446]

Example. Among $Wz = \{x \in X :$ is a least squares solution of $Lx=z\}$, we consider the problem of researching an approximation $\bar{x}$ of $w \in Wz$ such that $\| Aw-y \| \leqslant \| Ax-y \|$ for all $x \in Wz$. Let N(A) and N(L) be non-trivial subspaces and $N(A) \cap N(L) = \{0\}$ and $R(A_L)$ is closed.

[STEP 1] Take an initial approximation $x \in N(L)^\perp$, and $x_{n+1} = x_n - a_n r_n$ where $r_n = L^*Lx_n - L^*z, a_n = \| r_n \|^2 / \| Lr_n \|^2$. Then by Proposition 8

$$\| L^+z-x_n \| \leqslant C \left(\frac{M-m}{M+m} \right)^n$$

where C is a constant and M, m such that $m \| x \|^2 < (L^*Lx, x) < M \| x \|^2$ for all $x \in N(L)$. (see Groetsch[2] or Kantorvich[3])

[STEP 2] Let $\bar{y} = y-A(x_n)$, $y_{n+1} = y_n - a_n r_n$, $r_n = A^*{}_L A_L y - A^*{}_L \bar{y}$, $a_n = \| r_n \|^2 / \| A_L r_n \|^2$. Then

$$\| y_p - A^+{}_L \bar{y} \| \leqslant \frac{\|A_L\|^2 \, \|z_0-y^*\|^2 \, \|e_0\|^2}{\|A_L\|^2 \, \|z_0-y^*\|^2 + P\|e_0\|^2}$$

where $y^* = P_{R(A)}\bar{y}$, $A^*{}_L z_0 = y_0$, $e_0 = y_0 - A^+{}_L \bar{y}$.(see Groetsch [2])

[STEP 3] Take $\bar{x} = y_p + x_q$. Then

$$\| w-\bar{x} \| \leqslant \varepsilon_1 + (\| A^+{}_L \| \, \| A \| +1) \varepsilon_2 .$$

where

$$\varepsilon_1 = \frac{\|A_L\|^2 \|z_0 - y^*\|e_0\|^2}{\|A_L\|z_0 - y^*\|^2 + P\|e_0\|^2}$$

$$\varepsilon_2 = C\left\{\frac{M-m}{M+m}\right\}^q$$

$w \in Wz$ such that $\| Aw-y \| \leqslant \| Ax-y \|$ for all $x \in Wz$.

References

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