J. of Basic Sciences Cheju Nat. Univ.
5(1), 47~51, 1992.

기초과학연구 제주대학교
5(1), 47~51, 1992.

ENERGY MINIMIZING MAPS FROM ANNULUS TO S^2

GIE HYUN PARK

ABSTRACT. Bethuel, Brezis, Coleman and Helein considered energy minimizing maps from an annulus $\Omega_\rho = \{(x,y) \in \mathbb{R}^2 | \rho^2 \le x^2 + y^2 \le 1\}$ to sphere S^2 and showed that for $\rho > e^{-\pi}$, $u_0(x,y) = (\frac{x}{r}, \frac{y}{r}, 0)$ is the only minimizer, and that for $\rho \le e^{-\pi}$, there is a unique minimizer in the class of radially symmetric maps (or *radial* maps), and it differs from u_0 when $\rho < e^{-\pi}$. Sandier showed that a minimizer is actually radially symmetric. In this paper we present a more elementary and shorter proof.

1. Introduction

In this paper we consider energy minimizing maps from an annulus $\Omega_\rho = \{(x,y) \in \mathbb{R}^2 | \rho^2 \le x^2 + y^2 \le 1\}$ to the unit 2-sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$ among the admissible class $\mathcal{F}_\rho$ of maps that agree on the boundary $\partial\Omega_\rho$ with the map $u_0(x,y) = (\frac{x}{r}, \frac{y}{r}, 0)$, where $r = \sqrt{x^2 + y^2}$.

Recall that, roughly speaking, the energy functional E for a map u is defined to be

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2,$$

where Ω is the set on which u is defined.

We specify our admissible class $\mathcal{F}_\rho$ of maps as

$$\mathcal{F}_\rho = \{u \in H^1(\Omega_\rho, S^2) : u = u_0 \text{ at boundary } \partial\Omega_\rho\}$$

F. Bethuel, H. Brezis, B. D. Coleman and F. Helein [1] showed that

–For $\rho > e^{-\pi}$, $u_0(x,y) = (\frac{x}{r}, \frac{y}{r}, 0)$ is the only minimizer.

–For $\rho \le e^{-\pi}$, there is a unique minimizer in the class of radially symmetric maps (or *radial* maps), and it differs from u_0 when $\rho < e^{-\pi}$.

A radial map u is a map which is equivariant under the rotation of Ω_ρ round z-axis. For any rotation R, a radial map u satisfies $u \circ R(x,y) = R \circ u(x,y)$ so that

$$u(x,y) = (\frac{x}{r}\cos(\phi(r)), \frac{y}{r}\cos(\phi(r)), \sin(\phi(r))),$$

for some real valued function $\phi(r)$.

E. Sandier [2] proved that if u is a minimizer for the problem

$$\min_{v \in \mathcal{F}_\rho} \int_{\Omega_\rho} |\nabla v|^2,$$

then u is actually radially symmetric.

In this paper we will prove the above results in elementary ways and show that the unique solution is related to the simple pendulum, which can be stated as the following Main Theorem.

Theorem. *The minimizer is radially symmetric and*

–For $\rho > e^{-\pi}$, $u_0(x,y) = (\frac{x}{r}, \frac{y}{r}, 0)$ is the only minimizer.

–For $\rho \le e^{-\pi}$, there is a unique minimizer which comes from the pendulum equation.

2. Symmetry of minimizers

To prove the Main Theorem we need following Lemmas.

Lemma 1. *The functional*

$$W(\phi) = \frac{1}{2}\int_0^c \left(\left(\frac{d\phi}{ds} \right)^2 + \cos^2(\phi) \right) ds,$$

defined for maps ϕ with $\phi(0) = 0 = \phi(c)$, assumes its infimum at $\phi(s) = 0$ if $c < \pi$, otherwise at ϕ which is the angle function of a simple pendulum starting from the bottom and returns to it only at the time c.

Proof. The Euler-Lagrange equation is

$$2\frac{d^2\phi}{ds^2} + 2\cos(\phi)\sin(\phi) = 0, \tag{1}$$

This is the equation describing the motion of a simple pendulum. Behavior of its solutions is well-known and well-formulated in terms of elliptic functions of first type.

If $n\pi < c < (n+1)\pi$, there are $n+1$ solutions. First one is the constant function $\phi_0(s) = 0$. But this is not a minimizer of W if $\pi < c$. To see this we consider the perturbation $\phi_t(s) = t\sin(\frac{\pi}{c}s)$. The second derivative of the energy perturbation is

$$\begin{aligned}
&\frac{d^2}{dt^2}W(\phi_t)\\
=&\frac{d}{dt}\left\{\int_0^c t(\frac{\pi}{c})^2\cos^2(\frac{\pi}{c}s) - \cos(t\sin(\frac{\pi}{c}s))\sin(t\sin(\frac{\pi}{c}s))\sin(\frac{\pi}{c}s)\right\}\\
=&\int_0^c (\frac{\pi}{c})^2\cos^2(\frac{\pi}{c}s) + \sin^2(t\sin(\frac{\pi}{c}s))\sin^2(\frac{\pi}{c}s) - \cos^2(t\sin(\frac{\pi}{c}s))\sin^2(\frac{\pi}{c}s).
\end{aligned}$$

Thus

$$\left.\frac{d^2}{dt^2}W(\phi_t)\right|_{t=0} = \int_0^c (\frac{\pi}{c})^2\cos^2(\frac{\pi}{c}s) - \sin^2(\frac{\pi}{c}s) = ((\frac{\pi}{c})^2 - 1)\frac{c}{2}.$$

This is negative if $\pi < c$.

Other solutions ϕ vanishing at some interior points can not be a minimizer. We will define $\psi(s)$ which has less energy than ϕ. The graph of ϕ is similar to that of sine function. We flip it, i.e., we consider the graph of $|\phi|$. Erase between the first maximum point and the last one and connect two points by straight line. Let ψ be a function of which the graph is what we made.

Then clearly $W(\psi) < W(\phi)$.

The remaining possibility for a minimizer is what we want.

Lemma 2. *The functional*

$$F(\phi) = \int_0^{2\pi} (\phi'(\theta))^2 \, d\theta$$

defined for functions ϕ *with* $\phi(0) - \phi(2\pi) = 2\pi$ *assumes its infimum when* $\phi'(\theta) \equiv 1$.

Proof. Easy. I will not use the ink to elaborate this.

Proof of the Main Theorem. The energy functional in polar coordinate,

$$W(u) = \frac{1}{2} \int_\rho^1 \int_0^{2\pi} \left\{ \left| \frac{\partial u}{\partial r} \right|^2 + \frac{1}{r^2} \left| \frac{\partial u}{\partial \theta} \right|^2 \right\} r d\theta dr,$$

may be reduced to the following form by the change of variables $r = e^{-s}$

$$W(u) = \frac{1}{2} \int_0^{-\log \rho} \int_0^{2\pi} \left\{ \left| \frac{\partial u}{\partial s} \right|^2 + \left| \frac{\partial u}{\partial \theta} \right|^2 \right\} d\theta ds \tag{2}$$

We use the spherical coordinate for u, i.e. we put $u = (\cos\phi \cos\psi, \sin\phi \cos\psi, \sin\psi)$. This changes (2) to

$$\begin{aligned} &W(u) \\ =&\frac{1}{2} \int_0^{-\log \rho} \int_0^{2\pi} \left\{ \cos^2 \psi \left(\left(\frac{\partial \phi}{\partial s} \right)^2 + \left(\frac{\partial \phi}{\partial \theta} \right)^2 \right) + \left(\left(\frac{\partial \psi}{\partial \theta} \right)^2 + \left(\frac{\partial \psi}{\partial s} \right)^2 \right) \right\} d\theta ds \end{aligned}$$

Putting $f(s) = \max_\theta |\psi(s, \theta)|$ and using the Lemma 2, we obtain

$$\begin{aligned} &W(u) \\ \geq&\frac{1}{2} \int_0^{-\log \rho} \int_0^{2\pi} \left\{ \cos^2 f \left(\left(\frac{\partial \phi}{\partial s} \right)^2 + \left(\frac{\partial \phi}{\partial \theta} \right)^2 \right) + \left(\left(\frac{\partial \psi}{\partial \theta} \right)^2 + \left(\frac{\partial \psi}{\partial \theta} \right)^2 \right) \right\} d\theta ds \\ \geq&\frac{1}{2} \int_0^{-\log \rho} \int_0^{2\pi} \left\{ \cos^2 f + \left(\left(\frac{\partial \psi}{\partial \theta} \right)^2 + \left(\frac{\partial \psi}{\partial s} \right)^2 \right) \right\} d\theta ds \\ \geq&\frac{1}{2} \int_0^{-\log \rho} \int_0^{2\pi} \left\{ \cos^2 f + \left(\frac{\partial \psi}{\partial s} \right)^2 \right\} d\theta ds \end{aligned}$$

In the above the equality of the last holds if and only if $\frac{\partial \psi}{\partial \theta} \equiv 0$. In this case $\psi(s,\theta) = f(s)$, i.e., ψ is independent of θ. The second inequality holds if and only if $\phi(\theta) = \theta$. Thus if u is energy minimizing u is radially symmetric and in this case $W(u) = \frac{1}{2}\int_0^{-\log\rho}\int_0^{2\pi}\left\{\cos^2\psi + \left(\frac{\partial \psi}{\partial s}\right)^2\right\} d\theta ds$. Now Lemma 1 completes the proof.

References

1. Bethuel, F., Brezis, H., Coleman, B. D., and Hélein, F., *Bifurcation analysis of minimizing harmonic maps describing the equilibrium of nematic phases between cylinders*, Arch. Rational Mech. Anal. **118** (1992), 149–168.
2. Sandier, E., *Symétrie des applications harmoniques minimisantes d'une couronne vers la sphère*, C. R. Acad. Sci. Paris **313** (1991), 435–440.

Department of Mathematics
College of Natural Sciences
Cheju National University
Cheju 690 - 756
KOREA