J. of Basic Sciences, Cheju Nat. Univ.
9(1), 163~169, 1996

기초과학연구 제주대학교
9(1), 163~169, 1996

SPECTRAL MAPPING THEOREM AND WEYL'S THEOREM

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ABSTRACT. In this paper we give some conditions under which the Weyl spectrum of an operator satisfies the spectral mapping theorem for analytic functions. Also we show that Weyl's theorem holds for $p(T)$ where T is an operator of M-power class (N) and p is a polynomial on a neighborhood of $\sigma(T)$. Finally we answer an old question of Oberai.

0. Introduction

Throughout this paper let H denote an infinite dimensional Hilbert space and $B(H)$ the set of all bounded linear operators on H. If $T \in B(H)$, we write $\sigma(T)$ for the spectrum of T, $\pi_0(T)$ for the set of eigenvalues of T, $\pi_{0f}(T)$ for the set of eigenvalues of finite multiplicity, and $\pi_{00}(T)$ for the isolated points of $\sigma(T)$ that are eigenvalues of finite multiplicity. If K is a subset of $\mathbb{C}$, we write iso K for the set of isolated points of K. An operator $T \in B(H)$ is said to be *Fredholm* if its range ran T is closed and both the null space ker T and ker T^* are finite dimensional. The *index* of a Fredholm operator T, denoted by $i(T)$, is defined by

$$i(T) = \dim\ker T - \dim\ker T^*.$$

The *essential spectrum* of T, denoted by $\sigma_e(T)$, is defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not Fredholm}\}.$$

Received November 16, 1995. Revised June 2, 1996.
1991 AMS Subject Classification: 47A10, 47A53, 47B20.
Key words and phrases: Fredholm, Weyl, M-hyponormal, M-power class (N).
This paper was partially supported by Cheju National University Research Fund, 1996.

A Fredholm operator of index zero is called a *Weyl* operator. The *Weyl spectrum* of T, denoted by $\omega(T)$, is defined by

$$\omega(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not Weyl}\}.$$

It was shown ([1]) that for any operator T, $\sigma_e(T) \subset w(T) \subset \sigma(T)$, and $w(T)$ is a nonempty compact subset of $\mathbb{C}$.

Recall ([7]) that an operator $T \in B(H)$ is said to be *$M-$hyponormal* if there exists $M > 0$ such that

$$\|(T - z)^* x\| \le M\|(T - z)x\| \tag{1}$$

for all x in H and for all $z \in \mathbb{C}$.

Every hyponormal operator is $M-$hyponormal, but the converse is not true in general: for example, consider the weighted shift S on l_2 given by

$$S(x_1, x_2, \cdots) = (0, 2x_1, x_2, x_3, \cdots).$$

If T is both Fredholm and $M-$hyponormal, then $i(T) \le 0$. It was known that the mapping $T \to \omega(T)$ is upper semi-continuous, but not continuous at T([8]). However if $T_n \to T$ with $T_n T = T T_n$ for all $n \in \mathbf{N}$ then

$$\lim \omega(T_n) = \omega(T). \tag{2}$$

It was known that $\omega(T)$ satisfies the one-way spectral mapping theorem for analytic funcions: if f is analytic on a neighborhood of $\sigma(T)$ then

$$\omega(f(T)) \subset f(\omega(T)). \tag{3}$$

The inclusion (3) may be proper(see [2, Example 3.3]). If T is normal then $\sigma_e(T)$ and $\omega(T)$ coincide. Thus if T is normal and f is analytic on a neighborhood of $\sigma(T)$, it follows that $\omega(f(T)) = f(\omega(T))$ since $f(T)$ is also normal.

In this paper, we give some conditions under which the Weyl spectrum of an operator satisfies the spectral mapping theorem for analytic functions. Also we show that Weyl's theorem holds for $p(T)$ where T is an operator of M-power class (N) and p is a polynomial on a neighborhood of $\sigma(T)$. Finally we answer an old question of Oberai.

1. Spectral mapping theorem and Weyl's theorem

We recall ([2]) that for any operator $T \in B(H)$,

(4) $\sigma(T) - w(T) \subset \pi_{0f}(T)$ or equivaently $\sigma(T) - \pi_{0f}(T) \subset w(T)$.

THEOREM 1. *If either $\pi_{of}(T) = \phi$ or $\pi_{of}(T^*) = \phi$, then $w(f(T)) = f(w(T))$ for every analytic function f on a neighborhood of $\sigma(T)$.*

PROOF. Suppose $\pi_{of}(T) = \phi$. Since $\pi_{of}(p(S)) \subset p(\pi_{of}(S))$ for every operator S and any polynomial $p(t)$, $\pi_{of}(T) = \phi$ implies $\pi_{of}(p(T)) = \phi$. Therefore $w(T) = \sigma(T)$ and $w(p(T)) = \sigma(p(T))$ by (4). Since $\sigma(p(T)) = p(\sigma(T))$ by the usual polynomial spectral mapping formula,

$$w(p(T)) = \sigma(p(T)) = p(\sigma(T)) = p(w(T)).$$

Similarly if $\pi_{of}(T^*) = \phi$, then $w(p(T)) = p(w(T))$ since $w(T^*) = w(T)^*$.

If f is analytic on a neighborhood of $\sigma(T)$, then by Runge's theorem([4]), there is a sequence $(p_n(t))$ of polynomials converging uniformly in a neighborhood of $\sigma(T)$ to $f(t)$ so that $p_n(T) \to f(T)$. Since each $p_n(T)$ commutes with $f(T)$, by [8],

$$f(\omega(T)) = \lim p_n(\omega(T)) = \lim \omega(p_n(T)) = \omega(f(T)).$$

THEOREM 2. *If S and T are commuting M−hyponormal operators, then*

(5) $$S, T \text{ Weyl} \iff ST \text{ Weyl}.$$

PROOF. If S, T are Weyl, then S, T are Fredholm and $i(S) = i(T) = 0$. By [4], ST is Fredholm and by the index product theorem, $i(ST) = i(S) + i(T) = 0$. Hence ST is Weyl.

For the backward implication of (5) we note that if $ST = TS$, then $\ker S \cup \ker T \subseteq \ker ST$ and $\ker S^* \cup \ker T^* \subseteq \ker (ST)^*$. If ST is Weyl, then $\dim\ker S, \dim\ker T < \infty$ and $\dim\ker S^*, \dim\ker T^* < \infty$. Also ran S and ran T are closed by [6, Theorem 3.2.2]. Hence S, T are Fredholm.

Since S and T are $M-$hyponormal, $i(S) = i(T) = 0$ since $0 = i(ST) = i(S) + i(T)$.

If the "$M-$hyponormal" condition is dropped in the above Theorem, then the backward implication may fail even though T_1 and T_2 commute: For example, if U is the unilateral shift on l_2, consider the following operators on $l_2 \oplus l_2 : T_1 = U \oplus I$ and $T_2 = I \oplus U^*$.

Also we note that Theorem 2 holds for any hyponormal operator T since every hyponormal operator is $M-$hyponormal.

THEOREM 3. *If T is $M-$hyponormal and f is analytic on a neighborhood of $\sigma(T)$, then $\omega(f(T)) = f(\omega(T))$.*

PROOF. Suppose that $p(t)$ is any polynomial. Let

$$P(T) - \lambda I = a_0(T - \mu_1 I)\cdots(T - \mu_n I).$$

Since T is $M-$hyponormal, $T - \mu_i I$ are commuting $M-$hyponormal operators for each $i = 1, 2, \cdots, n$. It thus follows from Theorem 2 that

$$\begin{aligned}
\lambda \notin \omega(p(T)) &\Longleftrightarrow p(T) - \lambda I = \text{Weyl} \\
&\Longleftrightarrow a_0(T - \mu_1 I)\cdots(T - \mu_n I) = \text{Weyl} \\
&\Longleftrightarrow T - \mu_i I = \text{Weyl for each } i = 1, 2, \cdots, n \\
&\Longleftrightarrow \mu_i \notin \omega(T) \text{ for each } i = 1, 2, \cdots, n \\
&\Longleftrightarrow \lambda \notin p(\omega(T))
\end{aligned}$$

which says that $\omega(p(T)) = p(\omega(T))$.

If f is analytic on a neighborhood of $\sigma(T)$, then by Runge's theorem([4]), there is a sequence $(p_n(t))$ of polynomials converging uniformly in a neighborhood of $\sigma(T)$ to $f(t)$ so that $p_n(T) \to f(T)$. Since each $p_n(T)$ commutes with $f(T)$, by [8]

$$f(\omega(T)) = \lim p_n(\omega(T)) = \lim \omega(p_n(T)) = \omega(f(T)).$$

COROLLARY 4. *If T is hyponormal and f is analytic on a neighborhood of $\sigma(T)$, then $\omega(f(T)) = f(\omega(T))$.*

We say that *Weyl's theorem holds for T* if

$$\omega(T) = \sigma(T) - \pi_{00}(T).$$

There are several classes of operators including normal and hyponormal operators on a Hilbert space for which Weyl's theorem holds. Also it was shown in [8] that Weyl's theorem holds for any spectral operator of finite type on a Banach space. Oberai has raised the following question: Does there exist a hyponormal operator T such that Weyl's theorem does not hold for T^2? Note that T^2 may not be hyponormal even if T is hyponormal([5, Problem 209]). We will show that Weyl's theorem holds for $p(T)$ when T is an operator of M-power class (N). Thus we answer an old question of Oberai since every hyponormal operator is of 1-power class (N).

Recall ([9]) that $T \in B(H)$ is said to be *isoloid* if iso $\sigma(T) \subset \pi_0(T)$.

LEMMA 5. *([9]) Let $T \in B(H)$ be isoloid. Then for any polynomial $p(t)$, $p(\sigma(T) - \pi_{oo}(T)) = \sigma(p(T)) - \pi_{oo}(p(T))$.*

Let T be an $M-$hyponormal operator which satisfies the additional property that for all z in the complex plane, all integers n and all x in H,

$$\|(T-z)^n x\|^2 \leq M\|(T-z)^{2n}x\| \cdot \|x\|.$$

T is said to be an operator of ***$M-$power class*** (N)([7]). The following $M-$ hyponormal operator T which is not hyponormal is of $M-$power class (N) : Let $\{e_i\}$ be an orthonormal basis for H, and define

$$Te_i = \begin{cases} e_2, & \text{if } \ i = 1 \\ 2e_3, & \text{if } \ i = 2 \\ e_{i+1}, & \text{if } \ i \geq 3 \end{cases}$$

i.e., T is a weighted shift. From the definition of T we see that T is similar to the unilateral shift U([5], Problem 90). Thus there exists an operator S such that $T = SUS^{-1}$. In our case $\|S\| = 2$, $\|S^{-1}\| = 1$.

Since U is the unilateral shift, U is a hyponormal operator, and thus for every n and $z \in \mathbb{C}$ the operator $(U-z)^n$ is of class (N). It follows that

$$\|(U-z)^n x\|^2 \leq \|(U-z)^{2n} x\|$$

for all $x \in H$ with $\|x\| = 1$, and hence T is of M-power class with $M = 4$. Thus our class is strictly larger than the class of hyponormal operators.

THEOREM 6. *If $T \in B(H)$ is an operator of M-power class (N), then for any polynomial $p(t)$ Weyl's theorem holds for $p(T)$.*

PROOF. By [7], T is isoloid and Weyl's theorem holds for any operator of M-power class (N). Hence by Theorem 3 and Lemma 4,

$$w(p(T)) = p(w(T)) = p(\sigma(T) - \pi_{00}(T)) = \sigma(p(T)) - \pi_{00}(p(T))$$

Therefore Weyl's theorem holds for $p(T)$.

Since every hyponormal operator is of 1-power class (N), we obtain the following result which is answer for an old question of Oberai.

COROLLARY 7. *If $T \in B(H)$ is hyponormal, then for any polynomial $p(t)$ Weyl's theorem holds for $p(T)$.*

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