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FIBREWISE EXPONENTIAL LAWS IN SEQUENTIAL CONVERGENCE SPACES

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1. Introduction

In homotopy theory, in particular in the problem of fibration, the notion of exponential law plays central role. So many researchers have been tried to obtain convenient categories in which the exponential law exists [2-8,16,17]. So far, compactly generated spaces and quasi-topological spaces have been main objectives for the study from this point of view. However, in a structural point of view it has not been completely successful to find a convenient category of fibred spaces. The main reason was that the category of compactly generated spaces is not a quasitopos and quasi-topological spaces do not form a category, but a quasi-category. In 1986, J. Adamek and H. Herrlich showed that a topological category $\mathbf{C}$ is a quasitopos if and only if for any $B \in \mathbf{C}$, $\mathbf{C}_B$ is cartesian closed. Thus, it is natural to consider the category which is a quasitopos. With this consideration, in 1992, Min and Lee [13] obtained natural exponential laws in the category of convergence spaces over a base B.

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In first-countable topological spaces, one can restrict oneself to sequence in studying convergence and continuity. However, for more general spaces it seems to be assumed that sequences are not enough and that more general nets or filters must be used. But, it appears that in some senses sequences are adequate for all spaces considered up to now in analysis. Also, the main theorems of integration theory (dominated convergence, monotone convergence, etc.) are true only for sequences. The sequential language is useful as an alternative in metric spaces, and finally there is a fact that the convergent sequence and its limit form a compact set, while this is not true for nets. Thus there seems to be reason for direct study of sequential convergence. With this consideration, sequential convergence spaces have been studied from various points of view.

In this paper, we introduce sequential convergence spaces over a base space and construct a function space structure which will allow us fibrewise exponential laws.

2. Preliminaries

For any set X, let $X^{\mathbb{N}}$ be the set of all sequences on X. A ***sequential convergence space*** is an ordered pair (X, ξ) of sets, where $\xi \subseteq X^{\mathbb{N}} \times X$ is a specified relation between sequences $u \in X^{\mathbb{N}}$ and points $p \in X$ subject to the following three axioms:

(1) If $u_n = x$ for all n, then $((u_n), x) \in \xi$.

(2) If $((u_n), x) \in \xi$, then for every subsequence $u_{s(n)}$ of u_n, $(u_{s(n)}, x) \in \xi$.

(3) If $u \in X^{\mathbb{N}}$ is such that every subsequence $u_{s(n)}$ has a further subsequence $u_{st(n)}$ with $(u_{st(n)}, x) \in \xi$, then $((u_n), x) \in \xi$.

In what follows we will express the statement $((u_n), x) \in \xi$ by writing u_n converges to x in (X, ξ).

Let (X, ξ) and (Y, η) be sequential convergence spaces and $f : X \to Y$ be a map. Then f is called a *sequentially continuous map* if $f(u_n)$ converges to $f(x)$ in (Y, η) whenever u_n converges to x in (X, ξ).

The class of all sequential convergence spaces and sequentially continuous maps forms a category, which will be denoted by **Seq**.

Proposition 2.1. *[15]* **Seq** *has initial structures. The initial structure ξ induced by a family of functions $f_i : X \to Y_i (i \in I)$ and sequential convergence structures η_i on Y_i consists of precisely those pairs (u_n, x) such that for every $i \in I$ the sequence $f_i(u_n)$ converges to $f_i(x)$ in (Y, η_i).*

Proposition 2.2. *[15]* **Seq** *has final structures. The final structure η induced by an epimorphic family of functions $f_i : X_i \to Y (i \in I)$ and sequential structures ξ_i on X_i consists of precisely those pairs (v_n, y) such that for every subsequence $v_{s(n)}$ there exists a further subsequence expressible in the form $v_{st(n)} = f_i(u_n)$ for some choice of $i \in I$ and $u_n \in X_i$ such that u_n converges to x in (X_i, ξ_i) and $f_i(x) = y$.*

Let Y, Z be sequential convergence spaces and $C(Y, Z)$ be the set of all sequentially continuous maps from Y to Z. Then it is known that there is an external structure on $C(Y, Z)$ defined as follows: Give the final structure on $C(Y, Z)$ induced by the epimorphic family of functions $g : X \to C(Y, Z)$ for which the associated map $g^\dagger : X \times Y \to Z, g^\dagger(x, y) = g(x)(y)$ is a sequentially continuous map. Then we have the following result.

Theorem 2.3. *[15]* **Seq** *upholds an exponential law*

$$C(X \times Y, Z) \cong C(X, C(Y, Z)).$$

3. Function space structures

In this chapter, we define an internal structure on $C(Y, Z)$ and prove that this definition is equivalent to the external structure on $C(Y, Z)$. And, we introduce the notion of the sequential convergence space over a base space and define a function space structure which makes the category $\mathbf{Seq}_B$ to be cartesian closed.

For given sequential convergence spaces Y and Z, consider the following internal structure on $C(Y, Z)$.

Definition 3.1. The sequence f_n converges to f in $C(Y, Z)$ if for any subsequence $f_{s(n)}$ of f_n and any sequence y_n which converges to y in Y the sequence $f_{s(n)}(y_n)$ converges to $f(y)$ in Z.

Then we have the following result.

Proposition 3.2. *The above two definitions on $C(X, Y)$ are equivalent.*

Proof. Suppose f_n converges to f in $C(Y, Z)$ with respect to the internal structure. Let $f_{s(n)}$ be a subsequence of f_n and y_n be a sequence in Y converging to y. Let $E = \{y_n\} \cup \{y\}$. Define $g : E \to C(Y, Z)$ by $g(y_n) = f_{s(n)}$ and $g(y) = f$. Then for any other sequence x_n in Y converging to x, the sequence $g^\dagger(y_n, x_n) = g(y_n)(x_n) = f(s_n)(x_n)$ converges to $f(x) = g(y)(x)$ by the definition of the internal structure on $C(Y, Z)$. Therefore

$g^\dagger$ is sequentially continuous, and hence f_n converges to f in $C(Y, Z)$ with respect to the external structure.

Conversely, suppose f_n converges to f in $C(Y, Z)$ with respect to the external structure. Let $f_{s(n)}$ be a subsequence of f_n and y_n converge to y in Y. It remains to show that $f_{s(n)}(y_n)$ converges to $f(y)$ in Z, and hence it is enough to show that for any subsequence $f_{st(n)}(y_{t(n)}$ of $f_{s(n)}(y_n)$, there exists a further subsequence $f_{stu(n)}(y_{tu(n)})$ which converges to $f(y)$. By the definition of the external structure, there is a map $g : X \to C(Y, Z)$ and a sequence x_n converging to x in X such that $g(x_n) = f_{s(n)}, g(x) = f$ and $g^\dagger$ is sequentially continuous. But, since $(x_{tu(n)}, y_{tu(n)})$ converges to (x, y), the sequence $g^\dagger(x_{tu(n)}, y_{tu(n)}) = g(x_{(tu(n)})(y_{(tu(n)}) = f_{stu(n)}(y_{(tu(n)})$ converges to $g(x)(y) = f(y)$. Hence $f_{s(n)}(y_n)$ converges to $f(y)$ in Z, and so f_n converges to f in $C(Y, Z)$ with respect to the internal structure.

Now, consider the sequential convergence space over a base space.

For a given space B in **Seq**, the category $\mathbf{Seq}_B$ is defined as follows. An object in $\mathbf{Seq}_B$ is a pair (X, p) consisting of an object X of **Seq** and a morphism $p : X \to B$ of **Seq**. If (X, p) and (Y, q) are objects in $\mathbf{Seq}_B$, a morphism in $\mathbf{Seq}_B$ is a morphism $f : X \to Y$ of **Seq** such that $q \circ f = p$. In this case, X is called a *sequential convergence space over* B, p is called the *projection* and f is called a *sequentially continuous map over* B.

Proposition 3.3. $\mathbf{Seq}_B$ *has initial structures with respect to the family of functions* $f_i : X \to (Y_i, \eta_i)(i \in I)$.

Proposition 3.4. $\mathbf{Seq}_B$ *has final structures with repect to the epimorphic family of functions* $f_i : (X_i, \xi_i) \to Y (i \in I)$.

For given sequentially convergence spaces X and Y over B, let

$$C_B(X,Y) = \bigcup_{b \in B} C(X_b, Y_b)$$

as a set, where $C(X_b, Y_b)$ is the set of all sequentially continuous maps from X_b to Y_b. Define $((f_n), f) \in \xi$, where $\xi \subseteq C_B(X,Y)^{\mathbb{N}} \times C_B(X,Y)$ and $f \in C(X_b, Y_b)$ if

(1) let x_n converges to x in X with $x \in X_b$, then for any subsequence $f_{s(n)}$ of f_n, the sequence

$$f'_{s(n)}(x'_n) = \begin{cases} f_{s(n)}(x_n) & \text{if } f_{s(n)}(x_n) \text{ can be defined} \\ f(x) & \text{if not} \end{cases}$$

converges to $f(x)$ in Y,

(2) the sequence $p(f_n)$ converges to $p(f)$, where $p : C_B(X,Y) \to B$ is the projection defined by $p(g) = b$ for $g \in C(X_b, Y_b)$.

Proposition 3.5. $(C_B(X,Y), \xi)$ *is a sequential convergence space.*

Proof. Let $f_n = f$ for all n, and $f \in C(X_b, Y_b)$. Then if x_n converges to $x \in X_b$, for any subsequence $f_{s(n)}$ of f_n, the sequence

$$f'_{s(n)}(x'_n) = \begin{cases} f_{s(n)}(x_n) & \text{if } f_{s(n)}(x_n) \text{ can be defined} \\ f(x) & \text{if not} \end{cases}$$

is the image of a mixed sequence of a subsequence of x_n and a constant sequence x under f. Hence $f_{s(n)}(x_n)$ converges to $f(x)$ in Y. Trivially

$p(f_n) = b$ is a constant sequence in B, and hence $p(f_n)$ converges to $p(f)$. Therefore, $((f_n), f) \in \xi$.

Let f_n converges to f in $C_B(X, Y)$ and $f \in C(X_b, Y_b)$. Let $f_{s(n)}$ be a subsequence of f_n. We have to show that for any sequence x_n which converges to $x \in X_b$ in X, and for any subsequence $f_{st(n)}$ of $f_{s(n)}$, the sequence $f'_{st(n)}(x'_n)$ converges to $f(x)$. But, since f_n converges to f in $C_B(X, Y)$ and $f_{st(n)}$ is also a subsequence of f_n, $f'_{st(n)}(x'_n)$ converges to $f(x)$. And, since $p(f_{s(n)})$ is a subsequence of $p(f_n)$, $p(f_{s(n)})$ converges to $p(f)$. Therefore, $((f_{s(n)}), f) \in \xi$.

Let f_n be a sequence in $C_B(X, Y)$ such that any subsequence of f_n contains a further subsequence which converges to $f \in C(X_b, Y_b)$. We have to show that for any sequence x_n which converges to $x \in X_b$ in X, and any subsequence $f_{s(n)}$ of f_n, $f'_{s(n)}(x'_n)$ converges to $f(x)$ in Y. Since Y is a sequential convergence space, it is enough to show that for each subsequence $f'_{st(n)}(x'_{t(n)})$ of $f_{s(n)}(x_n)$, there is a further subsequence $f'_{stu(n)}(x'_{tu(n)}$ which convergea to $f(x)$. Note that $f_{st(n)}$ is a subsequence of f_n, and hence by assumption $f_{st(n)}$ has a further subsequence $f_{stv(n)}$ which converges to f. By the definition of ξ and the fact that $x_{tv(n)}$ converges to $x \in X_b$ in X, for any subsequence $f_{stvw(n)}$ of $f_{stv(n)}$, the sequence $f'_{stvw(n)}(x'_{tvw(n)})$ converges to $f(x)$ in Y. But, $f'_{stvw(n)}(x'_{tvw(n)})$ is also a subsequence of $f'_{st(n)}(x'_{t(n)})$. Hence $f'_{s(n)}(x'_n)$ converges to $f(x)$ in Y. Moreover, $p(f_n)$ converges to $p(f)$, since B is a sequential convergence space. Therefore, $((f_n), f) \in \xi$.

In all, $(C_B(X, Y), \xi)$ is a sequential convergence space.

Proposition 3.6. *The evaluation map* $ev : X \times_B C_B(X, Y) \to Y$ *defined by* $ev(x, f) = f(x)$ *is sequentially continuous.*

Proof. Let (x_n, f_n) be a sequence in $X \times_B C_B(X, Y)$ such that (x_n, f_n) converges to (x, f), where $x \in X_b$ and $f \in C(X_b, Y_b)$. Then x_n converges to x in X and f_n converges to f in $C_B(X, Y)$. Since f_n converges to f in $C_B(X, Y)$, for any subsequence $f_{s(n)}$ of f_n, the sequence

$$f'_{s(n)}(x'_n) = \begin{cases} f_{s(n)}(x_n) & \text{if } f_{s(n)}(x_n) \text{ can be defined} \\ f(x) & \text{if not} \end{cases}$$

converges to $f(x)$ in Y. Since f_n is also a subsequence of f_n,

$$f'_n(x'_n) = \begin{cases} f_n(x_n) & \text{if } f_n(x_n) \text{ can be defined} \\ f(x) & \text{if not} \end{cases}$$

converges to $f(x)$ in Y. But, since f_n and x_n are contained in the same fibre, this sequence is equal to $ev(x_n, f_n)$. Hence $ev(x_n, f_n)$ converges to $f(x) = ev(x, f)$ in Y. Therefore, ev is sequentially continuous.

Theorem 3.7. $\mathbf{Seq}_B$ *is cartesian closed.*

Proof. Let $f : X \times_B Z \to Y$ be a given sequentially continuous map. Define $\overline{f} : Z \to C_B(X, Y)$ by $\overline{f}(z)(x) = f(x, z)$ for $(x, z) \in X \times_B Z$. Let z_n converge to $z \in Z_b$ in Z. Then we have to show that $\overline{f}(z_n)$ converges to $\overline{f}(z)$ in $C_B(X, Y)$. Let x_n converge to $x \in X_b$ in X and $\overline{f}(z_{s(n)})$ be a subsequence of $\overline{f}(z_n)$. Then we have to show that the sequence

$$\overline{f}(z'_{s(n)})(x'_n) = \begin{cases} \overline{f}(z_{s(n)})(x_n) & \text{if } \overline{f}(z_{s(n)})(x_n) \text{ can be defined} \\ \overline{f}(z)(x) & \text{if not} \end{cases}$$

converges to $\overline{f}(z)(x)$ in Y. Consider the sequence $(\hat{x_n}, \hat{z_{s(n)}})$, where $(\hat{x_n}, \hat{z_{s(n)}}) = (x_n, z_{s(n)})$ if x_n and $z_{s(n)}$ are contained in the same fibre and $(\hat{x_n}, \hat{z_{s(n)}}) = (x, z)$ if not, which converges to (x, z). Then, since f is sequentially continuous, $f(x_n, z_{s(n)})$ converges to $f(x, z)$. But this sequence is equal to $\overline{f}(z'_{s(n)})(x'_n)$. Moreover, $p(\overline{f}(z_n))$ converges to $p(\overline{f}(z))$ in B. In all, $\overline{f}(z_n)$ converges to $\overline{f}(z)$ in $C_B(X, Y)$. Therefore, $\overline{f}$ is sequentially continuous.

Corollary 3.8. *For sequential convergence spaces X, Y and Z over B,*

$$\phi : C_B(X \times_B Y, Z) \to C_B(X, C_B(Y, Z))$$

is an isomorphism over B, where $\phi(f)(x)(y) = f(x, y)$.

For sequential convergence spaces X and Y over B, let $M_B(X, Y)$ the space of sequentially continuous maps from X to Y over B, equipped with the subspace structure of $C(X, Y)$ in **Seq**.

Proposition 3.9. *For sequential convergence spaces X and Y over B,*

$$\phi : M_B(X, Y) \to M_B(B, C_B(X, Y))$$

is an isomorphism over B, where $\phi(f)(b) = f_b : X_b \to Y_b$, the restriction of f on X_b.

Proof. Trivially, ϕ is bijective. Suppose that f_n converges to f in $M_B(X, Y)$, b_n converges to b in B and x_n converges to $x \in X_b$ in X. We have to show that for any subsequence $\phi(f_{s(n)})$ of $\phi(f_n)$, $\phi(f_{s(n)})(b_n)$ converges to $\phi(f)(b)$ in $C_B(X, Y)$, and hence that for any subsequence $\phi(f_{st(n)})(b_{t(n)})$, $\phi(f_{st(n)})(b_{t(n)})'(x_n')$ converges to $\phi(f)(b)(x)$ in Y. But, since f_n converges to

f in $M_B(X, Y)$, $f_{st(n)}(x_n)$ converges to $f(x)$. Note that $\phi(f_{st(n)})(b_{t(n)})'(x_n')$ is a mixed sequence of a subsequence $f_{st(n)}(x_n)$ of $f_{st(n)}(x_n)$ and a constant sequence $(f(x))$ and hence converges to $f(x) = \phi(f)(b)(x)$. Moreover, $r(\phi(f_{s(n)})(b_n))$ converges to $r(\phi(f)(b))$, since b_n converges to b, where $r : C_B(X, Y) \to B$ is the projection. Therefore, ϕ is continuous.

And, let $\phi^{-1} = \varphi$ and f_n converge to f in $M_B(B, C_B(X, Y))$. Suppose x_n converges to $x \in X_b$ and $x_n \in X_{b_n}$. Note that the projection $p : X \to B$ is sequentially continuous, and hence b_n converges to b. Since f_n converges to f in $M_B(B, C_B(X, Y))$, $f_{st(n)}(b_{t(n)})(x_{t(n)})$ converges to $f(b)(x)$. But, $\varphi(f_{s(n)}(x_n)) = f_{s(n)}(b_n)(x_n)$. This means that $\varphi(f_{s(n)})(x_n)$ contains a further subsequence which converges to $f(b)(x)$ and hence this sequence converges to $f(b)(x)$. Therefore, φ is continuous. In all, ϕ is an isomorphism.

Theorem 3.10. *For sequential convergence spaces X, Y and Z over B,*

$$\phi : M_B(X \times_B Y, Z) \to M_B(X, C_B(Y, Z))$$

is an isomorphism over B, where $\phi(f)(x)(y) = f(x, y)$.

Proof. It is easy to see that ϕ is a bijection. Suppose that f_n converges to f in $M_B(X \times_B Y, Z)$, x_n converges to $x \in X_b$ and y_n converges to $y \in Y_b$. We have to show that for any subsequence $\phi(f_{s(n)})$ of $\phi(f_n)$, $\phi(f_{s(n)})(x_n)$ converges to $\phi(f)(x)$, and hence that $\phi(f_{st(n)})(x_{t(n)})'(y_n')$ converges to $\phi(f)(x)(y)$. But, we note that $\phi(f_{st(n)})(x_{t(n)})'(y_n')$ is $\phi(f_{st(n)})(x_{t(n)})(y_n)$ if $x_{t(n)}$ and y_n are contained in the same fibre, and is $\phi(f)(x)(y)$ if not. Consider the sequence $(x_{t(n)}', y_n')$ in $X \times_B Y$ if $(x_{t(n)}', y_n')$ is $(x_{t(n)}, y_n)$ if $x_{t(n)}$ and y_n are contained in the same fibre, and is (x, y) if not. Then the sequence $\phi(f_{st(n)})(x_{t(n)})'(y_n')$

is equal to $f_{st(n)}(x_{t(n)}', y_n')$ which converges to $f(x, y) = \phi(f)(x)(y)$. Moreover, $r(\phi(f_{s(n)}))$ converges to $r(\phi(f))$, where $r : C_B(Y, Z) \to B$ is the projection. Hence ϕ is continuous.

Conversely, let $\phi^{-1} = \varphi$ and f_n converge to f in $M_B(X, C_B(Y, Z))$. Suppose (x_n, y_n) converges to (x, y) in $X \times_B Y$. Then $\varphi(f_n)(x_n, y_n) = f_n(x_n)(y_n)$, and hence $\varphi(f_n)(x_n, y_n)$ converges to $f(x)(y) = \varphi(f)(x, y)$. So, φ is continuous. In all ϕ is an isomorphism.

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