J. of Basic Sciences, Cheju Nat'l Univ.
19(2), 13~21, 2006

기초과학연구 제주대학교
19(2), 13~21, 2006

OPTIMALITY CONDITIONS FOR SOME OPTIMAL CONTROL PROBLEMS GOVERNED BY KELLER-SEGEL EQUATIONS

SANG-UK RYU

Department of Mathematics, Cheju National University,
Jeju 690-756, Korea

ABSTRACT. In this paper we study the non-smooth optimal control problems for the Keller-Segel equations presented by Keller and Segel(Ref. 5). We obtain the necessary conditions of optimality by introducing the approximating control problem.

2000 Mathematics Subject Classification: 49J20, 49J52
Key words and phrases: Keller-Segel equations, Non-smooth optimal control, Approximating control problem, Optimality conditions

1. INTRODUCTION

This paper is concerned with the optimal control problem

$$\underset{u}{\text{Minimize}} \quad J(u) \tag{P}$$

with the cost functional $J(u)$ of the form

$$J(u) = \int_0^T \|y(u) - y_d\|^2_{L^2(\Omega)} dt + \int_0^T l(u) dt, \quad u \in L^2(0,T; H^{1+\varepsilon}(\Omega)),$$

where $y = y(u)$ is governed by the Keller-Segel equations

$$\begin{cases} \partial_t y = a\Delta y - b\nabla \cdot \{y \nabla \rho\} & \text{in } \Omega \times (0,T], \\ \partial_t \rho = d\Delta\rho + fy - g\rho + \nu u & \text{in } \Omega \times (0,T], \\ \partial_n y = \partial_n \rho = 0 & \text{on } \partial\Omega \times (0,T], \\ y(x,0) = y_0(x), \quad \rho(x,0) = \rho_0(x) & \text{in } \Omega. \end{cases} \tag{K-S}$$

Here, Ω is a bounded region in $\mathbf{R}^2$ of C^3 class. ∂_t denotes the time derivative. a, b, d, f, g are given positive numbers and ν is a given nonnegative number. $u \geq 0$ is a control function varying in some bounded subset $\mathcal{U}_{ad}$ of $L^2(0,T; H^{1+\varepsilon}(\Omega))$, ε being some fixed exponent such that $0 < \varepsilon < 1/2$. $n = n(x)$ is the outer normal vector at a boundary point $x \in \partial\Omega$ and ∂_n denotes the differentiation along the vector n. $y_0(x)$ and $\rho_0(x)$ are nonnegative initial functions in $L^2(\Omega)$ and in $H^{1+\varepsilon}(\Omega)$, respectively. y, ρ are unknown functions of the Cauchy problem (K-S). Finally, l is a lower semicontinuous and convex function on $H^{1+\varepsilon}(\Omega)$.

The Keller-Segel equations (K-S) was introduced by Keller and Segel (Ref. 5) to describe the aggregating pattern formation of amoebae by chemotaxis. Unknown functions $y = y(x,t)$ and $\rho = \rho(x,t)$ denote the concentration of amoebae in Ω at time t and the concentration of chemical substance in Ω at time t, respectively. The chemotactic term $-b\nabla \cdot \{y\nabla\rho\}$ indicates that the cells are sensitive to chemicals and are attracted by them, and the production term fy indicates that the chemical substance is itself emitted by cells.

Many papers have already been published to study the control problems for nonlinear parabolic equations(Refs. 1, 2, 3, 6, and 7). In the recent paper (Ref. 8), Ryu and Yagi studied the optimal control problem for the Keller-Segel equations, that is, the existence of optimal controls and the necessary conditions of optimality were obtained by showing the differentiability of the cost functional with respect to the control. In the present paper we study the case which the differentiability is not guaranteed. In this sense, this paper may also be considered the generalization of Ref. 8 as the optimal control for a parabolic system of non-monotone type.

This paper is organized as follows. In Section 2, we shall formulate (K-S) as a semilinear equation in a product Hilbert space. Section 3 is devoted to obtaining the necessary conditions of optimality for (P) by introducing the approximating control problem.

2. The formulation of the problem

Let $A_1 = -a\Delta + a$ and $A_2 = -d\Delta + g$ be the Laplace operators equipped with the Neumann boundary conditions, $A_i (i = 1, 2)$ are linear isomorphisms from $H^1(\Omega)$ to $(H^1(\Omega))'$. As noticed in Ref. 9, $\mathcal{D}(A_i^\theta) = H^{2\theta}(\Omega)$ for $0 \le \theta < 3/4$, and $\mathcal{D}(A_i^\theta) = H_n^{2\theta}(\Omega)$ for $3/4 < \theta \le 3/2$.

We introduce two product Hilbert spaces $\mathcal{V} \subset \mathcal{H}$ as

$$\mathcal{V} = H^1(\Omega) \times \mathcal{D}(A_2^{1+\varepsilon/2}) \quad \text{and} \quad \mathcal{H} = L^2(\Omega) \times \mathcal{D}(A_2^{(1+\varepsilon)/2}),$$

respectively, where ε is some fixed exponent $\varepsilon \in (0, 1/2)$. By the identification of $\mathcal{H}$ and its dual $\mathcal{H}'$, we have: $\mathcal{V} \subset \mathcal{H} = \mathcal{H}' \subset \mathcal{V}'$. It is then seen that

$$\mathcal{V}' = (H^1(\Omega))' \times \mathcal{D}(A_2^{\varepsilon/2}).$$

We denote the scalar product of $\mathcal{H}$ by $(\cdot,\cdot)$ and the norm by $|\cdot|$. The duality product between $\mathcal{V}'$ and $\mathcal{V}$ which coincides with the scalar product of $\mathcal{H}$ on $\mathcal{H} \times \mathcal{H}$ is denoted by $\langle \cdot,\cdot \rangle$, and the norms of $\mathcal{V}$ and $\mathcal{V}'$ by $\|\cdot\|$ and $\|\cdot\|_*$, respectively.

We set also a symmetric sesquilinear form on $\mathcal{V} \times \mathcal{V}$:

$$a(Y, \tilde{Y}) = \left(A_1^{1/2} y, A_1^{1/2} \tilde{y}\right)_{L^2} + \left(A_2^{1+\varepsilon/2} \rho, A_2^{1+\varepsilon/2} \tilde{\rho}\right)_{L^2}, \qquad Y = \begin{pmatrix} y \\ \rho \end{pmatrix}, \tilde{Y} = \begin{pmatrix} \tilde{y} \\ \tilde{\rho} \end{pmatrix} \in \mathcal{V}.$$

Obviously, the form satisfies:

$$|a(Y,\widetilde{Y})| \le M\|Y\|\|\widetilde{Y}\|, \qquad Y,\widetilde{Y} \in \mathcal{V}, \tag{a.i}$$
$$a(Y,Y) \ge \delta\|Y\|^2, \qquad Y \in \mathcal{V} \tag{a.ii}$$

with some constants $M \ge 0$ and $\delta > 0$. This form then defines a linear isomorphism $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ from $\mathcal{V}$ to $\mathcal{V}'$, and the part of A in $\mathcal{H}$ is a positive definite self-adjoint operator in $\mathcal{H}$ with the domain $\mathcal{D}(A) = \mathcal{D}(A_1) \times \mathcal{D}(A_2^{(3+\epsilon)/2})$.

(K-S) is, then, formulated as an abstract equation

$$\begin{cases} Y' + AY = F(Y) + U(t), & 0 < t \le T, \\ Y(0) = Y_0 \end{cases} \tag{E}$$

in the space $\mathcal{V}'$. Here, Y' denotes the time derivative and $F(\cdot) : \mathcal{V} \to \mathcal{V}'$ is the mapping

$$F(Y) = \begin{pmatrix} -b\nabla \cdot \{y\nabla\rho\} + ay \\ fy \end{pmatrix}, \qquad Y = \begin{pmatrix} y \\ \rho \end{pmatrix} \in \mathcal{V}. \tag{2.1}$$

$U(t)$ and Y_0 are defined by $U(t) = \binom{0}{\nu u(t)}$ and $Y_0 = \binom{y_0}{\rho_0}$, respectively.

As verified in Refs. 8, $F(\cdot)$ satisfies the following conditions.

(f.i) For each $\eta > 0$, there exists an increasing continuous functions $\phi_\eta\colon [0,\infty) \to [0,\infty)$ such that

$$\|F(Y)\|_* \le \eta\|Y\| + \phi_\eta(|Y|), \quad Y \in \mathcal{V}.$$

(f.ii) For each $\eta > 0$, there exists an increasing continuous functions $\psi_\eta\colon [0,\infty) \to [0,\infty)$ such that

$$\|F(\widetilde{Y}) - F(Y)\|_* \le \eta\|\widetilde{Y} - Y\| + (\|\widetilde{Y}\| + \|Y\| + 1)\psi_\eta(|\widetilde{Y}| + |Y|)|\widetilde{Y} - Y|, \quad \widetilde{Y}, Y \in \mathcal{V}.$$

According to Theorem 2.1 of Ref. 8, we have the following result.

Theorem 2.1. *Let $0 \le y_0 \in L^2(\Omega)$, $0 \le \rho_0 \in H^{1+\epsilon}(\Omega)$, and let $0 \le u \in L^2(0,T;H^{1+\epsilon}(\Omega))$. Then,* (K-S) *possesses a unique nonnegative local solution*

$$0 \le y \in H^1(0,S;(H^1(\Omega))') \cap C([0,S];L^2(\Omega)) \cap L^2(0,S;H^1(\Omega)),$$
$$0 \le \rho \in H^1(0,S;H^\epsilon(\Omega)) \cap C([0,S];H^{1+\epsilon}(\Omega)) \cap L^2(0,S;H_n^{2+\epsilon}(\Omega)).$$

The time $S \in (0,T]$ is determined by the norms $\|u\|_{L^2(0,T);H^{1+\epsilon}(\Omega))}$, $\|y_0\|_{L^2(\Omega)}$ and $\|\rho_0\|_{H^{1+\epsilon}(\Omega)}$.

3 NECESSARY CONDITIONS OF OPTIMALITY

In this section, we obtain the necessary conditions of optimality for the Problem (P). We denote the scalar products in $\mathcal{V}$ and $\mathcal{V}'$ by $\langle\cdot,\cdot\rangle_{\mathcal{V}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{V}'}$, respectively. If we set $\mathcal{U} = L^2(0,T;\mathcal{H})$ and

$$\mathcal{U}_{ad} = \left\{ \begin{pmatrix} 0 \\ u \end{pmatrix} \in \mathcal{U};\ u \in L^2(0,T;H^{1+\epsilon}(\Omega)),\ u \geq 0,\ \|u\|_{L^2(0,T;H^{1+\epsilon})} \leq C \right\},$$

then $\mathcal{U}_{ad}$ is closed, bounded and convex subset of $\mathcal{U}$. The problem (P) is obviously formulated as follows

$$\operatorname*{Minimize}_{U \in \mathcal{U}_{ad}}\ J(U), \tag{$\overline{P}$}$$

where the cost functional $J(U)$ is of the form

$$J(U) = \int_0^T |DY(U) - Y_d|^2 dt + \int_0^T L(U) dt,\quad U \in \mathcal{U}_{ad}.$$

Here, $Y(U)$, $U \in \mathcal{U}_{ad}$, is the weak solution to (E) on a fixed interval $[0,T]$. $D : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator defined by $D\binom{y}{\rho} = \binom{y}{0}$ and $Y_d = \binom{y_d}{0}$ is a fixed element of $L^2(0,T;\mathcal{H})$ with $y_d \in L^2(0,T;L^2(\Omega))$. $L : \mathcal{H} \to \overline{\mathbb{R}}$ is a lower-semicontinuous and convex function defined by $L\binom{0}{u} = l(u)$.

It is verified in Ref. 8 that there exists an optimal control $\overline{U} \in \mathcal{U}_{ad}$ for $(\overline{P})$ such that $J(\overline{U}) = \min_{U \in \mathcal{U}_{ad}} J(U)$.

To derive the optimality conditions satisfied by an optimal control $\overline{U}$, the mapping $F(\cdot)$: $\mathcal{V} \to \mathcal{V}'$ defined by (2.1) must be Fréchet differentiable, and some estimate for the derivative $F'(Y)(\cdot)$ is necessary. In a direct calculations, $F(Y)$ is Fréchet differentiable with the derivative

$$F'(Y)Z = \begin{pmatrix} -b\nabla \cdot \{y\nabla w\} - b\nabla \cdot \{z\nabla \rho\} \\ fz \end{pmatrix}, Y = \begin{pmatrix} y \\ \rho \end{pmatrix},\ Z = \begin{pmatrix} z \\ w \end{pmatrix} \in \mathcal{V}.$$

Moreover, as verified in Refs. 8, the following properties are satisfied.

(f.iii) For each $\eta > 0$, there exists a constant $\mu_\eta, \nu : [0,\infty) \to [0,\infty)$ such that

$$|\langle F'(Y)Z, P\rangle| \leq \begin{cases} \eta\|Z\|\|P\| + C_\eta(\|Y\| + 1)\mu_\eta(|Y|)|Z|\|P\|, & Y,\ Z,\ P \in \mathcal{V}, \\ \eta\|Z\|\|P\| + C_\eta(\|Y\| + 1)\mu_\eta(|Y|)\|Z\||P|, & Y,\ Z,\ P \in \mathcal{V}, \\ \nu(|Y|)\|Z\|\|P\|, & Y,\ Z,\ P \in \mathcal{V}. \end{cases}$$

(f.iv) $F'(\cdot)$ is continuous from $\mathcal{H}$ into $\mathcal{L}(\mathcal{V},\mathcal{V}')$.

Let $\overline{U}$ be optimal control for the problem $(\overline{P})$ and $\overline{Y}$ is the solution to (E) with respect to $\overline{U}$. We consider the following approximate problem:

$$\operatorname*{Minimize}_{U}\ J_\epsilon(U) \tag{P_ϵ}$$

with the cost functional $J_\varepsilon(U)$ of the form

$$J_\varepsilon(U) = \int_0^T |DY(U) - Y_d|^2 dt + \int_0^T L_\varepsilon(U)dt + \int_0^T \{|Y - \overline{Y}|^2 + |U - \overline{U}|^2\}dt, \quad U \in \mathcal{U}_{ad}.$$

Here, $L_\varepsilon(\cdot) : \mathcal{H} \to \overline{\mathbf{R}}$ is defined by

$$L_\varepsilon(U) = \inf\{|U - V|^2/2\varepsilon + L(V); V \in \mathcal{H}\}. \tag{3.1}$$

First of all, we show the existence of the optimal solutions for (P_ε).

Lemma 3.1. *There exists an optimal control $U_\varepsilon \in \mathcal{U}_{ad}$ for (P_ε).*

Proof. As the proof is standard (cf. Refs 2 and 6), we will only sketch.

Let $\{U_n^\varepsilon\} \subset \mathcal{U}_{ad}$ be a minimizing sequence such that $\lim_{n\to\infty} J_\varepsilon(U_n^\varepsilon) = \min_{U\in\mathcal{U}_{ad}} J_\varepsilon(U)$. Since $\{U_n^\varepsilon\}$ is bounded, we can assume that $U_n^\varepsilon \to U_\varepsilon$ weakly in $L^2(0,T;\mathcal{H})$. For simplicity, we will write Y_n^ε instead of the solution $Y(U_n^\varepsilon)$ of (E) corresponding to U_n^ε,

$$\begin{cases} (Y_n^\varepsilon)' + AY_n^\varepsilon = F(Y_n^\varepsilon) + U_n^\varepsilon(t), & 0 < t \le T, \\ Y_n^\varepsilon(0) = Y_0. \end{cases}$$

As in the estimates of Theorem 2.1 in Ref. 8, we infer that the sequence $\{Y_n^\varepsilon\}$ is bounded in $L^2(0,T;\mathcal{V}) \cap H^1(0,T;\mathcal{V}')$. Therefore, choosing a subsequence if necessary, we can assume that

$$\begin{aligned} Y_n^\varepsilon \to Y_\varepsilon \quad &\text{weakly in} \quad L^2(0,T;\mathcal{V}), \\ (Y_n^\varepsilon)' \to (Y_\varepsilon)' \quad &\text{weakly in} \quad L^2(0,T;\mathcal{V}'). \end{aligned}$$

Since $\mathcal{V}$ is compactly embedded in $\mathcal{H}$, it is that

$$Y_n^\varepsilon \to Y_\varepsilon \text{ strongly in } L^2(0,T;\mathcal{H}). \tag{3.2}$$

Hence, by the standard argument, we infer that Y_ε is a solution to (E) with the control U_ε, that is, $Y_\varepsilon = Y(U_\varepsilon)$. On the other hand, (f.ii) implies that, for each $Z \in C([0,T];\mathcal{V})$,

$$\begin{aligned} &\int_0^T |\langle F(Y_n^\varepsilon(t)) - F(Y_\varepsilon), Z(t)\rangle| dt \\ &\le \int_0^T \{(\|Y_n^\varepsilon(t)\| + \|Y_\varepsilon\| + 1)\psi_\eta(|Y_n^\varepsilon(t)| + |Y_\varepsilon|)|Y_n^\varepsilon(t) - Y_\varepsilon|\|Z(t)\| \\ &\qquad + \eta\|Y_n^\varepsilon(t) - \overline{Y}(t)\|\|Z(t)\|\}dt = I_{1n} + I_{2n}. \end{aligned}$$

Then, Since $Y_n^\varepsilon \to Y_\varepsilon$ strongly $L^2(0,T;\mathcal{H})$ it follows that $\lim_{n\to\infty} I_1 n = 0$. Similarly, $\lim_{n\to\infty} \le C\eta\|Z\|_{L^2(0,T;\mathcal{V})}$. Since $\eta > 0$ is arbitrary, this shows that $F(Y_n^\varepsilon) \to F(Y_\varepsilon)$ strongly in $L^2(0,T;\mathcal{H})$. Hence, by the uniqueness, $\overline{Y}$ is the weak solution of (E) corresponding to $\overline{U}$. Since $Y_n^\varepsilon - Y_d \to Y_\varepsilon - Y_d$ strongly in $L^2(0,T;\mathcal{H})$ and (3.1) is lower semicontinuous, we have:

$$\min_{U\in\mathcal{U}_{ad}} J_\varepsilon(U) \le J_\varepsilon(U_\varepsilon) \le \varliminf_{n\to\infty} J_\varepsilon(U_n^\varepsilon) = \min_{U\in\mathcal{U}_{ad}} J_\varepsilon(U).$$

Hence, $J_\varepsilon(U_\varepsilon) = \min_{U\in\mathcal{U}_{ad}} J_\varepsilon(U)$. □

Lemma 3.2. *For $\varepsilon \to 0$, we have*

$$U_\varepsilon \to \overline{U} \quad \textit{strongly in } L^2(0,T;\mathcal{H}),$$
$$Y_\varepsilon \to \overline{Y} \quad \textit{strongly in } L^2(0,T;\mathcal{H}).$$

Proof. For any $\varepsilon > 0$, it follows from the inequality $L_\varepsilon(\overline{U}) \le L(\overline{U})$ that

$$J_\varepsilon(U_\varepsilon) \le J_\varepsilon(\overline{U}) \le J(\overline{U}).$$

Hence

$$\varlimsup_{\varepsilon\to 0} J_\varepsilon(U_\varepsilon) \le J((\overline{U}). \tag{3.3}$$

On the other hand, since $\{U_\varepsilon\}$ is bounded subset in $L^2(0,S;\mathcal{H})$, we assume that $U_\varepsilon \to U^*$ weakly in $L^2(0,T;\mathcal{H})$. Following the previous Lemma's proof, we see that $Y_\varepsilon \to Y^*$ strongly in $L^2(0,T;\mathcal{H})$ and $Y_\varepsilon \to Y^*$ weakly in $H^1(0,T;\mathcal{V}') \cap L^2(0,T;\mathcal{V})$. Since J_ε is lower semicontinuous in $L^2(0,T;\mathcal{H})$,

$$\varliminf_{\varepsilon\to 0} J_\varepsilon(U_\varepsilon) \ge J(U^*) \ge J(\overline{U})$$

and by (3.3)

$$\lim_{\varepsilon\to 0} \int_0^T |Y_\varepsilon - \overline{Y}|^2 + |U_\varepsilon - \overline{U}|^2 dt = 0.$$

Hence, we infer that $\overline{Y} = Y^*$, $\overline{U} = U^*$ and so the conclusions of Lemma 3.2. □

For the differentiability of $Y(U)$ with respect to U, we have the next statement (see Ref. 8).

Lemma 3.3. *Let* (a.i), (a.ii), (f.i), (f.ii), (f.iii), *and* (f.iv) *be satisfied. The mapping $Y : \mathcal{U}_{ad} \to H^1(0,T;\mathcal{V}') \cap C([0,T];\mathcal{H}) \cap L^2(0,T;\mathcal{V})$ is Gâteaux differentiable with respect to U. For $V \in \mathcal{U}_{ad}$, $Y'(U)V = Z$ is the unique solution in $H^1(0,T;\mathcal{V}') \cap C([0,T];\mathcal{H}) \cap L^2(0,T;\mathcal{V})$ of the problem*

$$\begin{cases} Z' + AZ - F'(Y)Z = V(t), & 0 < t \le T, \\ Z(0) = 0. \end{cases} \tag{3.4}$$

Lemma 3.4. *Let U_ε be an optimal control of (P_ε) and let $Y_\varepsilon \in L^2(0,T;\mathcal{V}) \cap \mathcal{C}([0,T];\mathcal{H}) \cap H^1(0,T;\mathcal{V}')$ be the optimal state; that is, Y_ε is the solution to* (E) *with the control U_ε. Then, there exists a unique solution $P_\varepsilon \in L^2(0,T;\mathcal{V}) \cap \mathcal{C}([0,T];\mathcal{H}) \cap H^1(0,T;\mathcal{V}')$ to the linear problem*

$$\begin{cases} -(P_\varepsilon)' + AP_\varepsilon - F'(Y_\varepsilon)^* P = DY_\varepsilon - Y_d + Y_\varepsilon - \overline{Y}, \quad 0 \le t < T, \\ P_\varepsilon(T) = 0 \end{cases} \tag{3.5}$$

in $\mathcal{V}'$. Moreover,

$$\int_0^T (\nabla L_\varepsilon(U_\varepsilon), V)dt + \int_0^T (P_\varepsilon, V)dt + \int_0^T (U_\varepsilon - \overline{U}, V)dt \ge 0 \qquad \forall V \in \mathcal{U}_{ad}. \tag{3.6}$$

Proof. From Lemma 3.3 and Gâteaux differentiable of L_ε, we see that J_ε is Gâteaux differentiable. It is well-known that

$$J_\varepsilon'(U_\varepsilon)(V - U_\varepsilon) \ge 0, \qquad \forall V \in \mathcal{U}_{ad}.$$

In a direct calculation, we have

$$\int_0^T (DY_\varepsilon - Y_d, Z_\varepsilon)dt + \int_0^T (\nabla L_\varepsilon(U_\varepsilon), V)dt + \int_0^T (Y_\varepsilon - \overline{Y}, Z_\varepsilon)dt \ge \int_0^T (\overline{U} - U_\varepsilon, V)dt \quad \forall V \in \mathcal{U}_{ad}, \tag{3.7}$$

where Z_ε is the solution to

$$\begin{cases} (Z_\varepsilon)' + AZ_\varepsilon - F'(Y_\varepsilon)Z_\varepsilon = V(t), \qquad 0 < t \le T, \\ Z_\varepsilon(0) = 0. \end{cases}$$

Let P_ε be a solution of the equation (3.5). From Chap. XVIII, Theorem 2 of Ref. 4, we know that there exists a unique solution $P_\varepsilon \in L^2(0,S;\mathcal{V}) \cap \mathcal{C}([0,T];\mathcal{H}) \cap H^1(0,T;\mathcal{V}')$ of (3.5). If we multiply (3.5) by Z_ε and integrate on $(0,T)$, we have

$$\begin{aligned} &\int_0^T (DY_\varepsilon - Y_d + Y_\varepsilon - \overline{Y}, Z_\varepsilon)dt \\ &= \int_0^T (P_\varepsilon, (Z_\varepsilon)' + AZ_\varepsilon - F'(Y_\varepsilon)Z_\varepsilon)dt \\ &= \int_0^T (P_\varepsilon, V)dt. \end{aligned}$$

Hence, we get, by (3.7),

$$\int_0^T (\nabla L_\varepsilon(U_\varepsilon), V)dt + \int_0^T (P_\varepsilon, V)dt + \int_0^T (U_\varepsilon - \overline{U}, V)dt \ge 0 \qquad \forall V \in \mathcal{U}_{ad}. \quad \square$$

Now we want to make ε tend to 0. We need further estimates on P_ε.

Lemma 3.5. *When $\varepsilon \to 0$,*

$$P_\varepsilon \to \overline{P} \quad \textit{weakly in } L^2(0,T;\mathcal{V}) \cap H^1(0,T;\mathcal{V}'),$$
$$P_\varepsilon \to \overline{P} \quad \textit{strongly in } L^2(0,T;\mathcal{H}).$$

Here, $\overline{P}$ is the solution of the linear problem

$$\begin{cases} -P' + AP - F'(\overline{Y})^* P = D\overline{Y} - Y_d, & 0 \le t < T, \\ P(T) = 0 \end{cases} \tag{3.8}$$

in $\mathcal{V}'$.

Proof. By the standard argument, we infer that P_ε is bounded in $L^2(0,T;\mathcal{V}) \cap H^1(0,T;\mathcal{V}')$. Therefore, we can obtain that

$$P_\varepsilon \to P^* \quad \text{weakly in } L^2(0,T;\mathcal{V}) \cap H^1(0,T;\mathcal{V}'),$$
$$P_\varepsilon \to P^* \quad \text{strongly in } L^2(0,T;\mathcal{H}).$$

Let us verify that P^* is the solution of (3.8). In fact, it suffice to show that

$$[F'(Y_\varepsilon)]^* P_\varepsilon \to [F'(\overline{Y})]^* P^* \quad \text{weakly in } L^2(0,T;\mathcal{V}'). \tag{3.9}$$

(f.iii) implies that, for each $Z \in C([0,T];\mathcal{V})$,

$$\begin{aligned} &\int_0^T \langle [F'(Y_\varepsilon)]^* P_\varepsilon - [F'(\overline{Y})]^* P^*, Z \rangle dt \\ &\quad \le \int_0^T \eta \|P_\varepsilon - P\| \|Z\| dt + C_\eta \int_0^T (\|Y_\varepsilon\| + 1)\mu_\eta(|Y_\varepsilon|) |P_\varepsilon - P^*| \|Z\| dt \\ &\qquad + \int_0^T \|[F'(Y_\varepsilon)]^* P^* - [F'(\overline{Y})]^* P^*\|_* \|Z\| dt = I_{1\varepsilon} + I_{2\varepsilon} + I_{3\varepsilon}. \end{aligned}$$

Since $P_\varepsilon \to P^*$ strongly in $L^2(0,T;\mathcal{H})$, we have

$$\lim_{\varepsilon \to 0} I_{2\varepsilon} = 0.$$

Moreover, since $Y_\varepsilon \to \overline{Y}$ strongly in $\mathcal{H}$ a.e. t, it follows from (f.iv) that

$$[F'(Y_\varepsilon)]^* P^* \to [F'(\overline{Y})]^* P^* \text{ strongly in } \mathcal{V}' \text{ a.e..}$$

By the dominated convergence theorem,

$$\lim_{\varepsilon \to 0} I_{3\varepsilon} = 0.$$

For $I_{1\varepsilon}$,

$$\lim_{\varepsilon \to 0} I_{1\varepsilon} \le C\eta \|Z\|_{L^2(0,T;\mathcal{V})}.$$

Since $\eta > 0$ is arbitrary, this shows that (3.9) is satisfied. Hence, by the uniqueness, we infer $\overline{P} = P^*$. □

With the aid of the previous Lemmas, we can easily show the necessary conditions of optimality.

Theorem 3.6. *Let $\overline{U}$ be an optimal control of $(\overline{P})$ and let $\overline{Y} \in L^2(0,T;\mathcal{V}) \cap \mathcal{C}([0,T];\mathcal{H}) \cap H^1(0,T;\mathcal{V}')$ be the optimal state. Then, there exists a unique solution $P \in L^2(0,T;\mathcal{V}) \cap \mathcal{C}([0,T];\mathcal{H}) \cap H^1(0,T;\mathcal{V}')$ to the linear problem*

$$\begin{cases} -P' + AP - F'(\overline{Y})^* P = D\overline{Y} - Y_d, & 0 \le t < T, \\ P(T) = 0 \end{cases} \tag{3.10}$$

in $\mathcal{V}'$. Moreover,

$$-P(t) \in \partial L(\overline{U}) \qquad a.e.\ t \in (0,T).$$

Here, ∂L denotes the subdifferential of L.

Proof. Letting $\varepsilon \to 0$ in (3.5), it follows by Lemma 3.5 that there exists a unique solution $P \in L^2(0,T;\mathcal{V}) \cap \mathcal{C}([0,T];\mathcal{H}) \cap H^1(0,T;\mathcal{V}')$ of (3.10). By Lemma 3.2, since $U_\varepsilon \to \overline{U}$ in $L^2(0,T;\mathcal{H})$, it follows from a standard argument (Ref. 2) that

$$\int_0^T (\nabla L_\varepsilon(U_\varepsilon), V)dt \to \int_0^T (\xi, V)dt \quad \text{with } \xi(t) \in \partial L(\overline{U}) \text{ a.e. } t \in (0,T) \tag{3.11}$$

for all $V \in \mathcal{U}_{ad}$. Letting $\varepsilon \to 0$ in (3.6), it follows from Lemma 3.5 and (3.11) that

$$\int_0^T (P,V)dt + \int_0^T (\xi, V)dt \ge 0 \quad \forall V \in \mathcal{U}_{ad}. \quad \square$$

References

1. Ahmed, N. U., and Teo, K. L., *Optimal Control of Distributed Parameter Systems*, North-Holland, New York, 1981.
2. Barbu, V., *Analysis and Control of Nonlinear Infinite Dimensional Systems*, Academic Press, Boston, 1993.
3. Casas, E., Fernández, L. A., and Yong, J., *Optimal control of quasilinear parabolic equations*, Proceedings of royal society of edinburgh Section A, Vol. 125, pp. 545–565, 1995.
4. Dautray, R., and Lions, J. L., *Mathematical Analysis and Numerical Methods for Science and Technology*, Vol. 5, Springer-Verlag, Berlin, 1992.
5. Keller, E. F., and Segel, L. A., *Initiation of slime mold aggregation viewed as an instability*, J. Theor. Biol., Vol. 26, pp. 399–415, 1970.
6. Lions, J. L., *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, Berlin, 1971.
7. Papageorgiou, N. S., *On the optimal control of strongly nonlinear evolution equations*, Journal of Mathematics Analysis and Applications, Vol. 164, pp. 83–103, 1992.
8. Ryu, S.-U., and Yagi, A., *Optimal control of Keller-Segel equations*, Journal of Mathematics Analysis and Applications, Vol. 256, pp. 45–66, 2001.
9. Triebel, H., *Interpolation theory, function spaces, differential operators*, North-Holland, Amsterdam, 1978.