濟州大學校 產業技術硏究所 論文集, 第10卷, 2號, pp.174-180, 1999. 12
Cheju Nat'l Univ. Res. Insti. Ind.Tech. Jour., Vol.10, No.2 pp.174-180, 1999. 12

The Onset of Natural Convection in Horizontal Fluid Layer Heated Uniformly from Below

Min-Chan Kim* and Myung-Taek Hyun**

밑면이 일정 열속으로 가열되는 수평 유체층에서의 자연대류의 발생

김 민 찬* · 현 명 택**

ABSTRACT

The critical condition of the onset of buoyancy-driven convective motion of uniformly heated horizontal fluid layer was analysed by the propagation theory. The onset time is obtained as a function of the Rayleigh number and the Prandtl number. Our theoretical results predicted the experimental results, quite reasonably.

Key Words : Buoyancy effect, stability analysis, propagation theory

Ⅰ. Introduction

When an initially quiescent fluid layer is heated from below with a certain Rayleigh number exceeding critical value, the buoyancy-driven convective motion occurs. This convective motion driven by buoyancy forces has attracted many researcher's attention from the beginning of this century. It is well-known that buoyancy-driven convection plays an important roles in many engineering problem, such as chemical vapor deposition, solidification, electroplating and also many other conventional processes involving heat and mass transfer. Most of these processes involves non-linear, developing temperature profiles and therefore it is one of the most important problem to predict when or from where the buoyancy-driven motion sets in.

Choi et al.[1] proposed the propagation theory to analyse the buoyancy-driven convection phenomena. In their analysis, they introduce the thermal-boundary layer thickness as a new length scaling factor and transformed disturbance equations similarly under the linear stability theory principle of exchange of stabilities. In propagation theory, the onset conditions are defined as the conditions that the fastest growing disturbances start to grow rapidly. Their predicted results compared with experimental data of initially quiescent horizontal

* 제주대학교 화학공학과, 산업기술연구소
Dept. of Chem. Eng. & Res. Inst. Ind. Tech., Cheju Nat'l Univ.
** 제주대학교 기계공학과, 산업기술연구소
Dept. of Mech. Eng. & Res. Inst. Ind. Tech., Cheju Nat'l Univ.

fluid layers[2], initially quiescent fluid-saturated horizontal porous fluid layers[3], laminar-forced convection flow[4], and laminar-natural convection flow[5], reasonably well.

In this study we consider the buoyancy effects in horizontal fluid layer heated from below. Here will be analysed the onset condition of buoyancy-driven convective motion and compared the predicted value with available experimental data.

II. Stability Analysis

2.1. Governing Equations

The system considered here is a Newtonian fluid with an initial temperature T_i confined by two infinite parallel plates. The fluid layer of depth "d" is heated from below with constant flux q_w. The upper boundary is kept at initial temperature T_i. The schematic diagram of the base system is shown in Fig. 1. For this system the governing equations of flow and temperature fields are expressed by employing the Boussinesq approximation, as follows:

$$\nabla \cdot \vec{U}=0 \tag{1}$$

$$\left\{\frac{\partial}{\partial t}+\vec{U}\cdot\nabla\right\}\vec{U}=\frac{1}{\rho_r}\nabla P+\nu\nabla^2\vec{U}+g\beta T\vec{k} \tag{2}$$

$$\left\{\frac{\partial}{\partial t}+\vec{U}\cdot\right\}T=\alpha\nabla^2 T \tag{3}$$

where $\vec{U}$, T, P, μ, α, g, ρ and β represent the velocity vector, the temperature, the pressure, the viscosity, the thermal diffusivity, the gravitational acceleration, the density, and the thermal expansion coefficient, respectively. The subscript "r" represents the reference state.

The important parameters to describe the present system are the Prandtl number Pr and the Rayleigh number based on the bottom heat flux Ra_q defined

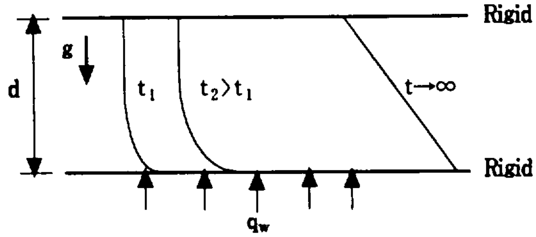

Fig. 1 Schematic diagram of base system

by

$$Pr=\frac{\nu}{\alpha} \quad \text{and} \quad Ra_q=\frac{g\beta q_w d^4}{k\alpha\nu} \tag{4}$$

where k and ν denote the thermal conductivity and the kinematic viscosity, respectively. Ra_q is sometimes called the dimensionless heat flux. In case of slow heating the basic temperature profile is linear and time-independent and its critical condition is independent of Pr and represented by

$$Ra_{q,c}=1296 \tag{5}$$

But for a rapid heating system of large Ra_q, the stability problem becomes transient and complicated, and the critical time t_c to mark the onset of buoyancy-driven motion remains unsolved.

For the conduction state the base temperature field can be governed by the following dimensionless forms:

$$\frac{\partial\theta_0}{\partial\tau}=\frac{\partial^2\theta_0}{\partial z^2} \tag{6}$$

with the following initial and boundary conditions.

$$\theta_0=0 \quad \text{at} \quad \tau=0 \text{ and } \quad z=1 \tag{7.a}$$

$$\frac{\partial\theta_0}{\partial z}=-1 \quad \text{at} \quad z=0 \tag{7.b}$$

where $\tau=d^2/(\alpha t)$, $z=Z/d$ and $\theta_0=k(T-T_i)/q_w d$. The subscript 0 denote the base state. The Graetztype solution of base temperature field can be obtained by employing conventional separation of variable technique as follows:

$$\theta_0 = 1 - z - 2\sum_{n=1}^{\infty}\frac{1}{\mu_n}\cos(\mu_n z)\exp(-\mu_n^2\tau) \quad (8)$$

where $\mu_n = (n-1/2)\pi$. For deep-pool systems, the Leveque-type solution can be obtained as follows[6]:

$$\theta_0 = \sqrt{\frac{4\tau}{\pi}}\left\{\exp\left(-\frac{\zeta^2}{4}\right) - \zeta erfc\left(\frac{\zeta}{2}\right)\right\} \quad (9)$$

where $\zeta = z/\sqrt{\tau}$. The above equation is in good agreement with the exact solution (8) in the region of ＜ 0.1.

Since we are primarily concerned with the deep-pool case of large Ra_q and small τ, the above Leveque type solution (9) represents the basic temperature profile quite well. Although the above Leveque-type solution represent the base temperature profile, for the mathematical convenience we introduce the dimensionless variable θ_0^*:

$$\theta_0^* = \frac{\theta_0}{\sqrt{\tau}} \quad (10)$$

Then the base temperature field within $\tau \leq 0.1$ can be transformed to

$$\frac{d^2\theta_0^*}{d\zeta^2} + \frac{\zeta}{2}\frac{d\theta_0^*}{d\zeta} - \frac{1}{2}\theta_0^* = 0 \quad (11)$$

with the boundary conditions

$$\frac{d\theta_0^*(0)}{d\zeta} = 0 \quad \text{and} \quad \theta_0^*(\infty) = 0 \quad (12)$$

The solution of equation (11) satisfying equation (12) can be obtained by conventional numerical scheme and is the same as the base temperature profile of equation (9).

2.2. Stability Equations

Under the linear stability theory disturbances caused by the onset of thermal convection can be formulated, in dimensionless form, in terms of the temperature component θ_1 and the vertical velocity component w_1 by transforming equations (1) ~ (3):

$$\left\{\frac{1}{Pr}\frac{\partial}{\partial\tau} - \nabla^2\right\}\nabla^2 w_1 = -\nabla_1^2\theta_1 \quad (13)$$

$$\frac{\partial\theta_1}{\partial\tau} + Ra_q w_1\frac{\partial\theta_0}{\partial z} = \nabla_1^2\theta_1 \quad (14)$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ and $\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Here the velocity component has the scale of α/d and the temperature component has the scale of $\alpha\nu/(g\beta d^3)$. The proper boundary conditions are given by

$$w_1 = \frac{\partial w_1}{\partial z} = \frac{\partial\theta_1}{\partial z} = 0 \quad \text{at} \quad z = 0 \quad (15.a)$$

$$w_1 = \frac{\partial w_1}{\partial z} = \theta_1 = 0 \quad \text{at} \quad z = 1 \quad (15.b)$$

Our goal is to find the critical time τ_c for a given Pr and Ra_q by using equations (13) ~ (15).

Based on the normal mode analysis, convective motion is assumed to exhibit the horizontal periodicity. Then the perturbed quantities can be expressed as follows:

$$[w_1(\tau,x,y,z), \theta_1(\tau,x,y,z)] = [w_1(\tau,z), \theta(\tau,z)]\exp[i(a_x x + a_y y)] \quad (16)$$

where "i" is the imaginary number. The horizontal wave number "a" has the relation of $a = [a_x^2 + a_y^2]^{1/2}$. Based on the scaling relation, the following relation is assumed:

$$\left|\frac{w_1}{\theta_1}\right| \sim \delta_T^2 \tag{17}$$

where $\delta_T(\propto\sqrt{\tau})$ is the dimensionless thermal boundary layer thickness.

With the above reasoning the dimensionless amplitude functions of most dangerous mode are assumed to have the form of

$$[w_1(\tau,z),\theta(\tau,z)]=[\tau w_1^*(\zeta),\theta_1^*(\zeta)] \tag{18}$$

By using these relations the stability equation is obtained from the equations (13) and (14) as

$$\left\{(D^2-a^{*2})^2+\frac{1}{2Pr}(\zeta D^3-a^{*2}\zeta D+2a^{*2})\right\}w^*=-a^{*2}\theta^* \tag{19}$$

$$\left(D^2+\frac{1}{2}\zeta D-a^{*2}\right)\theta^*=Ra^*w^*D\theta_0^* \tag{20}$$

where $a^*=a\sqrt{\tau}$, $Ra^*=Ra_q\tau^2$ and $D=d/d\zeta$. For the deep-pool case, the boundary conditions, equation (15), are transformed as follows:

$$w^*=Dw^*=D\theta^*=0 \text{ at } \zeta=0 \tag{21.a}$$

$$w^*=Dw^*=\theta^*=0 \text{ as } \zeta\to\infty \tag{21.b}$$

It is assumed that a^* and Ra^* are the eigenvalues, and also the onset time of buoyancy-driven convection for a given Ra_q is unique under the principle of exchange of stabilities. The above procedure is the essence of our propagation theory. Our propagation theory relaxed frozen-time model by considering the terms involving $\partial(\cdot)/\partial\tau$ in equations (13) and (14).

2.3. Stability Analysis Results

In the limiting case of infinite or zero Prandtl number, the governing equations are reduced to simpler form because the inertia or viscous terms are negligible, respectively. For this limiting case, Lee et al.[7] analysed the stability conditions. They approximate base temperature profile and disturbances distributions by using integral method and WKB approximation. For infinite Prandtl number case, our results are compared with Lee et al.'s reults in Table 1. It shows good agreements between their critical conditions and ours. This means that our numerical scheme is quite favorable to analyse the stability equations.

Table 1 Comparison our critical conditin with Lee et al.s for the liminting case of Pr

	$Pr\to\infty$	$Pr\to 0$
Present Study	20.03	-
Lee et al.	20.88	8.59/Pr

For finite Prandtl numbers, the critical values of a_c^* and Ra_c^* are summarized in Table 2. Based on these results and Lee et al.'s result for $Pr\to 0$, the critical condition can be represented as:

$$Ra_c^*=20.03\left[1+\left(\frac{0.43}{Pr}\right)^{2/3}\right]^{3/2} \tag{22}$$

It seems evident that Ra_c^* increase with decrease in Pr, and the Pr effect on critical conditions is negligible for $Pr\geq 10$. The Pr effect becomes pronounced for $Pr<1$. This means that the inertia terms make system more stable. Very viscous liquids

Table 2 Numerical values of critical conditions for the various Pr

Pr	0.01	0.1	0.7	1	7	10	100	∞
Ra_c^*	1122.30	158.64	45.90	39.04	23.36	22.41	20.29	20.03
a_c^*	0.73	0.73	0.67	0.66	0.57	0.56	0.52	0.52

are used in experiments of Nielsen and Sabersky[8], and Chu[9]. The Prandtl numbers of the fluids used in their experiments are 45~4700 and 4×10^5. As mentioned above, the critical conditions are nearly independent of Pr for Pr≥10, so we adopt the infinite Pr case as the basis of the comparison between theoretical and experimental results. For infinite Pr case, the stability criteria can be expressed as follows:

$$\tau_c = 4.47 Ra_q^{-1/2} \quad \text{and} \quad a_c = 0.25 Ra_q^{1/4} \qquad (23)$$

The above results is compared with the experimental data of Nielsen and Sabersky[8] and the theoretical results of Kim and Kim[10] in Fig. 2. As shown in Fig. 2, our τ_c is lower than the experimental data, however Kim and Kim's results shows fairly good agreement with experimental data. This discrepancy is due to the difference in the definition of critical condition of each study. We define τ_c as the time that infinitesimal disturbances start to grow exponentially, but Kim and Kim[10] as the time that the Nusselt number is increase 1% with respect to

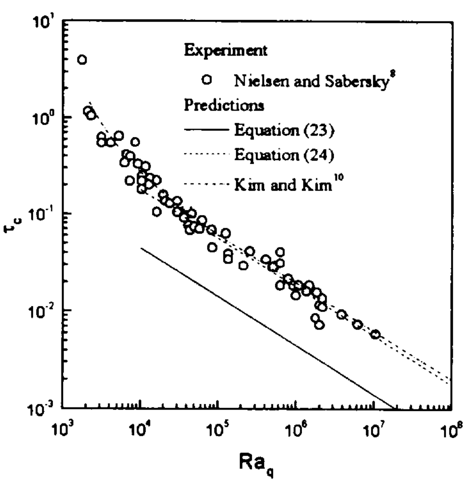

Fig. 2 Comparison of critical conditions with Nielsen and Sarbersky's data[8]

that of conduction state. It can be assumed that a certain time is required after the onset of disturbances to amplify the disturbances to affect the Nusselt number. This may explain the difference between our critical time and Kim and Kim's.

Foster[11] proposed that the onset time of natural convection obtained by using the thermal boundary layer thickness as a length scaling factor should be too short by factor of 4. By accepting Foster's concept, we suggest that the disturbances set in at τ_c will lead to manifest convection at $4\tau_c$. Thus, it is assumed that the onset time when the convective motion can be detectable experimentally, τ_o can be given as follows:

$$\tau_o = 17.92 Ra_q^{-1/2} \qquad (24)$$

The above relation is compared with Nielsen and Sabersky's experimental works in Fig. 2.

Another experiments were conducted by Chu[9]. He represents his experimental data as Ra vs. Ra_q plot. For convenience of comparison, we reconstruct our stability condition by using the following relation obtained from base temperature profile

$$\frac{k\Delta T}{q_w d} = \frac{2}{\sqrt{\pi}}\sqrt{\tau} \qquad (25)$$

as

$$Ra_c = 2.39 Ra_q^{-3/4} \qquad (26)$$

where Ra_c is the critical Rayleigh number based on temperature difference between two plates. In Fig. 3, Chu's experimental results are compared with the theoretical results of ours and Kim and Kim's[10].

As shown in Fig. 3, our critical condition represents the incipient motion criteria fairly well, whereas Kim and Kim's results show good agreement with $Nu_{\min}$ criteria. It is assumed that incipient motion criteria have nearly same physical meaning of the present τ_c, and Nu_{min} criteria are related with critical condition of Kim and Kim's. From the Chu's visualization results, we can obtain very important information on the growth of disturbances. When the horizontal fluid layer is heated from below, a certain time is required to make the buoyancy-driven convection set in. Once the disturbances set in, they grow continuously and affect the Nusselt number. The Nusselt number follows conduction state to a certain time after onset of disturbances, and deviates from conduction state. And further time is required to show minimum point and undershoot.

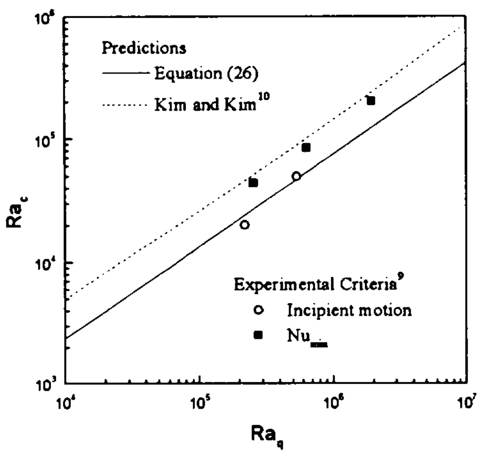

Fig. 3 Comparison with critical conditions with Chu's data[9)]

Chu's Nu_{min} criteria have good agreements with Nilsen and Sabersky's results. From this, it can be assumed that Nielsen and Sabersky's criteria correspond to Chu's Nu_{min} criteria. From these, it seems evident that the disturbances set in at the present τ_c, and grow enough to show the minimum point of the Nusselt number around $4\tau_c$ and that slightly after $4\tau_c$, the Nusselt number increase 1% with respect to conduction state.

Ⅳ. Conclusion

The critical condition of the onset of buoyancy-driven motion of uniformly heated horizontal fluid layer has been analysed by the propagation theory. It is interesting that our theoretical predictions have close agreement with experimental results. Therefore, it may be stated that our propagation theory is a powerful tools to examine the buoyancy-driven phenomena in horizontal fluid layers.

요 약

밑면이 일정 열속으로 가열되는 수평 유체층에서 부력에 의한 자연대류 발생 시점을 전파이론을 적용하여 해석하였다. 자연대류 발생시점을 Rayleigh 수와 Prandtl 수의 함수로 구하였다. 본 연구의 해석결과는 기존의 실험결과를 잘 설명하여 준다.

References

1) Choi, C.K., Shin, C.B. and Hwang, S.T., 1986, Thermal instability in thermal entrance region of Couette flow heated uniformly from below, Proc. 8th Int. Heat Transfer Conf., San Francisco, Vol. 3, pp.1389-1394.

2) Kim, M.C., Choi, K.H. and Choi, C.K., 1999a, The onset of thermal convection in initially, stably stratified fluid layer, *Int. J. Heat Mass Transfer*, Vol.42, pp.4253-4258.

3) Yoon, D.Y. and Choi, C.K., 1989, Thermal convection in a staturated porous medium subjected to isothermal heating, *Korean J. Chem. Eng.*, Vol.6, pp.144-149.

4) Kim, M.C., Baik, J.S., Hwang, I.G., Yoon, D.Y., Choi, C.K., 1999b, Buoyancy-driven convection in plane Poiseuille flow. *Chem. Eng. Sci.*, Vol.54,

pp.619-632.

5) Chun, Y.H. and Choi, C.K., 1991, Thermal instability of natural convection over inclined isothermal heated plates, *Hwahak Konghak*, Vol. 29, pp.381-387.

6) Lee J.D., Choi, C.K. and Yoon, D.Y., 1988, An anlaysis of thermal convection in a horizontal fluid layer heated uniformly from below, Proc. 1st KSME-JSME Thermal and Fluid Eng. Conf., Vol.2, pp.342-347.

7) Carslaw, H.S. and Jaeger, J.C., 1959, Conduction of heat in solid, 2nd ed., Oxford.

8) Nielsen, R.C. and Sabersky, R.H., 1973, Transient heat transfer in Benard Convection, *Int. J. Heat Mass Transfer*, Vol.16, pp.2407-2420.

9) Chu, T.Y., Developing convection above a finite horizontal surface, Proc. 9th Int. Heat Transfer Conf., Jerusalem, Vol.2, pp.169-174.

10) Kim, K.-H. and Kim, M.-U., 1986, The onset of natural convection in a fluid layer suddenly heated from below, *Int. J. Heat Mass Transfer*, Vol.29, pp.193-201.

11) Foster, T.D., 1969, Onset of manifest convection in a layer of fluid with a time-dependent surface temperature, *Phys. Fluids*, Vol.12, pp. 2482-2487.