" The best invariant estimator for variance of normal distribution "

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概 要

平均이 주어진 正規分布의 分散 σ^2의 推定量의 決定問題를 parameter space Θ와 action space $\mathcal{a}$가 모두 R이고 loss function $L(\sigma^2, a)$가 $(\ln a - \ln \sigma^2)^2$인 statistical decision problem $(\Theta, \mathcal{a}, L)$로 보고 sufficiency와 invariance의 두 原理를 적용하여 해결하였다.

I Introduction

In a statistical decision problem, a admissible rule in the class of invariant rules is also admissible within the class of all decision rules. A best decision rule (that is, a rule that is as good as other rule) therefore in the class of invariant rules is still a best one in the class of all rules. And a best invariant rule may be admissible for many decision problems. Moreover the class of decision rules based on a sufficient statistic forms an essentially complete class. In the present work, we shall find the best decision rule in the class of invariant rules for a specific decision problem, estimator for variance of normal distribution on the basis of a sample by means of the principles of sufficiency and invariance both.

II Preliminaries

The decision problems of estimating a parameter θ by a best invariant rule, in which θ is a location parameter of the distribution of the observable random variable X, with loss function is a function of $(a-\theta)$ alone, $L(\theta,a)=L(a-\theta)$, is founded on the following theorem 1 (see Ferguson [3])

Theorem 1; In the problem of estimating a location parameter with loss $L(\theta,a)=L(a-\theta)$, if $E_0L(X-b)$ exists and is finite for some b and if there exists a b_0 such that

$$E_0L(X-b_0)=\inf_b E_0L(X-b),$$

where the infimum is taken over all b for which $E_0L(X-b)$ exists, then $d(x)=x-b$. is a best invariant rule.

Theorem 2 (a) θ is a scale parmeter for the distribution of random variable X if, and only if, the distribution of X/θ when θ is the true value of the parameter is independent of θ.(b) If the distributions of X are absolutely continuous with probability density function $f_X(x;\theta)$, then θ is a scale parameter for the distribution of X if, and only if, $f_X(x;\theta)=\left(\frac{1}{\theta}\right)f\left(\frac{x}{\theta}\right)$ for some density $f(x)$.

The proof of theorem 2 is not hard and we omit the proof (see Wilk [5]).

III Main result

Consider the problem of the estimating the variance σ^2 of a normal distribution with zero mean on the basis of a sample of size n, Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with zero mean and unknowing variance σ^2. The new random variable $Z=\sum_{i=1}^{n} X_i^2$, square sum of the ramdom sample, is sufficient statistics for σ^2. First we take the random variable Z as observbale random variable in the decision problem for the sake of using the principle of sufficiency. The variance of normal distribution is not location parameter, but scale parameter of distribution of random variable Z because the distribution of $\frac{Z}{\sigma^2}$ has a chi-sguare distribution with n degree of freedom, namely gamma distribution $G(\frac{n}{2}, 2)$, independent σ^2. Therefore the problem of estimating the parameter σ^2 of normal distribution with zero mean on the basis of a sample can be considered a statistical decision problem (θ, a, L) with an observable random quantity Z whose distribution depends on scale parameter σ^2, where the parameter space θ and action space ov are both real line, the loss function $L(\sigma^2, a)=(\ln a\text{-}\ln\sigma^2)^2$.

Next transform the scale parameter problem to a location parameter problem by the tranformation $g(z)=\ln z$ in order to apply the theorem 1. The finding the best invariant rule for the scale parameter σ^2 in original problem is equivalent to estimating a location parameter $\ln\sigma^2$ for the distribution of random variable Z when loss function is $L(\ln\sigma^2, \ln a)= L(e^{\ln\sigma^2}, e^{\ln a})$. Therefore the theorem 1 indicates that the best invariant estimate of $\ln\sigma^2$ for this new problem is $\ln z\text{-}b_o$, where b_o is the value of b which minimizes

$$\begin{aligned}&E[L\ (\ln z-b)\,|\ln\sigma^2=0]\\ &=E[L\ (\ln z-b)\,|\ \sigma^2=1]\\ &=E[\ (\ln z-b)^2\ |\ \sigma^2=1]\end{aligned}$$

therefore,

$$b^o=[\ln z|\sigma^2=1]=\frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}\int_o^\infty \ln Z.Z^{\frac{n}{2}-1}e^{-\frac{z}{2}}dz$$

Putting 2t=z in above integral gives

$$b_o=H(n)+\ln 2$$

where $H(n)$ represent the following **integral value,**

$$H(n)=\frac{1}{\Gamma(\frac{n}{2})}\int_o^\infty \ln t - t^{\frac{n}{2}-1}\cdot e^{-t}\,dt$$

Since the bestinvariant estimator of $\ln\sigma^2$ is $\ln z\text{-}(H(n)+\ln 2)$, the best invariant estimator of σ^2 is $\frac{Z}{2e^{H(n)}}$. Moreover we can find that $2e^{H(n)}$ is $(n-1)$ approximately, so that the best invariant estimator of σ^2 using the loss $L(\sigma^2, a)=(\ln a-\ln\sigma^2)^2$ is close to $\frac{Z}{n-1}$,

that is $\frac{1}{n-1}\sum_{i=1}^{n} X_i^2$

References

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