On the Range of a Vector Valued Measure

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Vector値 測度의 値域에 관하여

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ABSTRACT

This paper gives a sufficient condition in order that the range of a vector valued measure be precompact. It is just that the average range of a Banach space valued measure on a measurable set X_i with a finite measure is precompact. And also it gives the some properties of measurable functions using the definition of the essential range of a measurable function.

1. Introduction

The first striking theorem on the range of a vector valued measure was Liapounoff's theorem appeared in 1940 which says that the range of a measure with values in a finite dimesional vector space is compact. In 1968 Rieffel generalized the Radon-Nikodym theorem to vector valued measures employing the Bochner integral. In 1969 Uhl showed that a vector valued measure with bounded variation whose values are either in a reflexive space or a separable dual space has a precompact range. In 1973 Cho T. and Tong A. extended Rieffel's Radon-Nikodym theorem and Uhl's result on the range of a Banach space valued measure.

The purpose of this note is to find an another sufficient condition in order that the range of a vector valued measure be precompact. In addition to this, we can show the some propertis of the measurable functions using the definition of the essential range of a measurable function.

2. Measurable function

Let (X, Σ, μ) be a σ-finite measure space and let B be a Banach space. We use the following definition. A B-valued function, f, on X is measurable if it is the pointwise limit a.e. of a sequence of B-valued simple measurable functions.

Definition 2.1. Let f be a measurable function, and let $E \in \Sigma$. Then the essential range of f restricted to E, $er_E(f)$, is defined to be the set of those $b \in B$ such that for every $\varepsilon > 0$ the measure of $\{x \in E : \|f(x) - b\| < \varepsilon\}$ is strictly positive.

Proposition2.2. If f is a measurable function, and if $E \in \Sigma$, then

(a) If $\mu(E) = 0$, then $er_E(f) = \phi$;

(b) If $\mu(E) > 0$, then $er_E(f) \neq \phi$.

Proof. (a) If $er_E(f) \neq \phi$, then there exists $b \in B$ such that $\mu\{x \in E : \|f(x) - b\| < \varepsilon\} > 0$ by the Definition of $er_E(f)$. Hence $\mu(E) > 0$.

(b) Assume, without loss of generality, that $f(E)$ is separable. Suppose that $er_E(f)=\phi$. Then $er_E(f)\cap f(E)=\phi$, and thus for eah $x\in E$ there exists an $\varepsilon_x>0$ such that

$$\mu(\{y\in E: \|f(y)-f(x)\|<\varepsilon_x\})=0.$$

Therefore

$$f(E)\subset\bigcup_{x\in E}B\varepsilon_x(f(x)),$$

the open balls center $f(x)$ and radius ε_x. Since $f(E)$ is separable, there exists a countable subcollection of those open balls $B\varepsilon_{x_n}(f(x_n))$ with $f(E)\subset\bigcup_{n\in N}B\varepsilon_{x_n}(f(x_n))$.

Then $E\subset\bigcup_{n\in N}\{y\in E: \|f(y)-f(x_n)\|<\varepsilon_{x_n}\}$, so that $\mu(E)=0$.

Proposition2.3. If f is a measurable function, then f is locally almost essentially compact valued (i.e., given $E\in\Sigma$ with $\mu(E)<\infty$, and given $\varepsilon>0$, there is an $F\in\Sigma$, $F\subset E$ such that $\mu(E-F)<\varepsilon$ and $er_F(f)$ is compact).

Proof. Since f is a measurable function, so let $\{f_n\}$ be a sequence of simple measurable functions converging to f a.e.. By Egoroff's theorem Dunford N. and Schwartz J. T. 1958, f_n converges to f almost uniformly on E (i. e., there is an $F\in\Sigma$, $F\subset E$ such that $\mu(E-F)<\varepsilon$ and f_n converges to f uniformly on F). Since $er_F(f)=\{b\in B: \mu\{x\in F: \|f(x)-b\|<\varepsilon\}>0\}$, so let $\{b_1\cdots, b_k\}=\text{Range}(f)$. Then $er_F(f)\subset\bigcup_{i=1}^{k}B_\varepsilon(b_i)$, and so $er_F(f)$ is totally bounded.

3. The range of a vector valued measure

Let X be a point set and Σ be a σ-field of subsers of X. If B is a Banach space, then B-valued measure is a countably additive set function F defined on Σ with values in B. Let (X,Σ,μ) be a σ-finite measure space, then there exists a sequence $\{X_i\}$ of sets in Σ such that $X=\bigcup_{i=1}^{\infty}X_i$ with $\mu(X_i)<\infty$. Define the average range of F on X_i is

$$A_{X_i}(F)=\{\frac{F(N_i)}{\mu(N_i)}: N_i\subset X_i,\quad N_i\in\Sigma, 0<\mu(N_i)\}.$$

And F is of bounded variation if

$$\text{var}(F)(X)=\sup_{\Pi}\sum_n\|F(E_n)\|<\infty$$

where the supremum is taken over all partitions $\Pi=\{E_n\}_{n=1}^{m}\subset\Sigma$ consisting of a finite collection of disjoint sets in Σ whose union is X. Here, we can restate the main theorem of Rieffel M. A. 1968 as followings:

Lemma 3.1. Let (X,Σ,μ) be a σ-finite measure space and let F be a B-valued measure on Σ where B is a Banach space. Then F is the indefinite integral with respect to μ of a Bochner integrable function $f:X\to B$ if and only if

(1) $F\ll\mu$(i. e., F is absolutely continuous with respect μ on Σ),

(2) F is of bounded variation,

(3) locally F somewhere has compact average range (i. e., $A_{X_i}(F)$ is (norm) compact).

It is shown in Uhl J. J., Jr. 1968 that a sufficient condition in order that the range of F be precompact is that the Banach space B is either a reflexive space or a separable dual space. Here we give a sufficient condition in order that the range of F be precompact if the condition that the Banach spacs B is reflexive or a separable dual is omitted.

Theorem 3.2. Let (X,Σ,μ) be a σ-finite measure space. If $A_{X_i}(F)$ is precompact, then the range of F is precompact.

Proof. Let the operator $T: L^1(X,\Sigma,\mu)\longrightarrow B$ be a linear extension of F shch that $T(\alpha\chi_M+\beta\chi_N)=\alpha F(M)+\beta F(N)$ for characteristic functions χ_M, χ_N, and $M\in\Sigma$, $N\in\Sigma$. Since $A_{X_i}(F)$ is precompact, so T is locally compact (i.e., the restriction of the operator T to $L^1(X_i, \Sigma, \mu)$ is compact for each i). Since any sequence of measurable subsets of X can be rewritten by a disjoint sequence of measurable sets, so, without loss of generality, we may assume that $\{X_i\}$ is a disjoint one. Therefore, by an inductive application of the Dunford-Pettis-Phillips theorem (Dunford N. and Pettis B.J. 1940, Phillips R. S. 1943), there exits a Bochner integrable function f: $X\longrightarrow B$ such that $T(g)=\int gf\,d\mu$ for each $g\in L^1(X,\Sigma,\mu)$. Now select a sequence $\{\chi_n\}$ of simple functions with their values in B converging to f. Define $T_n, n=1,2,\cdots$, by $T_n(g)=\int_X g\chi_n d\mu X_x$ for $g\in L^1(X, \Sigma, \mu)$. Then the range of each T_n is finite dimensional since each χ_n is a simple function. Thus each T_n is a compact operator. Here T_n and T are bounded since

$$\|T_n(g)\|\leq\int_X|g|\,\|\chi_n\|d\mu\leq\|g\|_{L_1}\|\chi_n\|_{L_\infty}$$ by

Hölder's inequality. And

$$\lim_{n\to\infty}\|T_n-T\|\leq\lim_{n\to\infty}\int\|\chi_n-f\|d\mu=0$$

since $\chi_n\longrightarrow f$. Therefore T is compact operator since T_n is compact. Hence the range of F is precompact since $T(\alpha\chi_M+\beta\chi_N)=\alpha F(M)+\beta F(N)$.

Remark. The hypothesis of the **Theorem 3.2.** is weaker than that of Uhl's results Uhl J.J., Jr 1969. That is, this theorem extends Uhl's results. Here the hypothesis of **Theorem3.2** is just the (3) of **Lemma 3.1.**

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要 略

本 論文에서는 σ-finite 測度 空間 (X,Σ,μ)의 σ-field Σ로부터 Banach 空間 B로 가는 Vector値 測度 F의 Range가 Precompact이기 위한 充分 條件을 調査하는데 그 目的이 있다. 이 充分 條件은 有限 測度를 갖는 Measurable 集合 X_i 위에서 定意된 Vector値 F의 Average Range $A_{X_i}(F)$가 Precompact임을 보였다. 또한 Measurable 函數의 性質을 이 函數의 Essential Range의 定意로부터 調査하였다.