ON THE NONHOLONOMIC COMPONENTS OF THE CHRISTOFFEL SYMBOLS IN V_n (I)

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V_n 공간에서 Christoffel symbol의 Non-holonomic components에 관하여 (I)

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Summary

In this paper we study the inverse of the results obfained in our previous paper; Hyun, J.O & Kim, H.G 1980(on the christo ffel symbols of the non-holonomic frames in V_n) in order to reconstruct and to investigate the useful relationships between holononomic and nonholonomic components of the christoffel symbols.

1. INTRODUCTION.

Let $h_{\lambda\mu}$ be the fundamental metric tensor, whose determinant

(1.1) $h \overset{\text{def}}{=\!=} \det(h_{\lambda\mu}) \neq 0$

and let $\underset{i}{e}^{\nu}$ $(i=1,2,\ldots,n)$ be a set of n linearly independent vectors in n-dimensional Riemannian space V_n refered to a real coordinate system x^{ν}.

Then there is a unique tensor $h^{\lambda\nu}=h^{\nu\lambda}$ defined by

(1.2) $h_{\lambda\mu}\, h^{\lambda\nu} \overset{\text{def}}{=\!=} \delta^{\nu}_{\mu}$

and a unique reciprocal set of n linearly independent covariant vectors $\overset{i}{e}_{\lambda}$ $(i=1,2,\ldots,n,)$, satisfying

(1.3)* $\underset{i}{e}^{\nu}\, \overset{i}{e}_{\lambda}=\delta^{\nu}_{\lambda}\ ,\ \ \underset{j}{e}^{\lambda}\, \overset{i}{e}_{\lambda}=\delta^{i}_{j}.$

Within the vectors $\underset{i}{e}^{\nu}$ and $\overset{i}{e}_{\lambda}$ a nonholonomic frame of V_n defined in the following way;

If $T^{\nu\cdots}_{\lambda\cdots}$ are holonomic components of a tensor, then its nonholonomic components are defined by

(1.4) a $\quad T^{i\cdots}_{j\cdots} \overset{\text{def}}{=\!=} T^{\nu\cdots}_{\lambda\cdots}\, \overset{i}{e}_{\nu}\, \underset{j}{e}^{\lambda}\cdots$

From (1.3) and (1.4) a

(1.4) b $\quad T^{\nu\cdots}_{\lambda\cdots} \overset{\text{def}}{=\!=} T^{i\cdots}_{j\cdots}\, \underset{i}{e}^{\nu}\, \overset{j}{e}_{\lambda}\cdots.$

In this paper, for our further discussion, results obtained in our previous paper Chung, K.T & Hyun, J.O 1976 and Hyun, J.O & Kim, H.G 1980 will be introduced without proof.

2. PRELIMINARY RESULTS.

Theorem (2.1). We have

(*) Throughout the present paper, all indices take the values $1,2,\ldots,n$ and follow the summation convention. Greek indices are used for the holonomic components of a tensor, while Roman indices are used for the nonholonomic components of a tensor.

(2.1) a $\qquad \underset{i}{e}^{\nu}=\overset{j}{e}_{\lambda}\, h_{ij}\, h^{\lambda\nu}, \quad \overset{j}{e}_{\lambda}=\underset{i}{e}^{\nu}\, h^{ij}\, h_{\lambda\nu}$

(2.1) b $\qquad h_{ij}=\delta_{ij}, \; h^{ij}=\delta^{ij}, \quad \underset{i}{e}^{\nu}=\overset{i}{e}^{\nu}, \quad \overset{j}{e}_{\lambda}=\underset{j}{e}_{\lambda}.$

Consider a symmetric covariant tensor a whose determinant $a \overset{\text{def}}{=\!=} \det(a_{\lambda\mu})\neq 0$. It is well-known that the quantities defined by

$$a^{\lambda\nu} \overset{\text{def}}{=\!=} \frac{\text{cofector of } a_{\lambda\nu} \text{ in } a}{a}$$

is a symmetric contravariant tensor satisfying

(2.2) $\qquad a_{\lambda\mu}\, a^{\lambda\nu}=\delta_{\mu}^{\nu}$

Theorem (2.3). The holonomic and nonholonomic components of the christoffel symbols satisfy

(2.3) a $\qquad [jk,m]_a=[\lambda\mu,\omega]_a\, \underset{j}{e}^{\lambda}\, \underset{k}{e}^{\mu}\, \underset{m}{e}^{\omega} + a_{\lambda\mu}(\partial_{\gamma}\, \underset{j}{e}^{\lambda})\, \underset{k}{e}^{\gamma}\, \underset{m}{e}^{\mu},$

(2.3) b $\qquad \left\{ {i \atop jk} \right\}_a = -\left\{ {\nu \atop \lambda\mu} \right\}_a \overset{i}{e}_{\nu}\, \underset{j}{e}^{\lambda}\, \underset{k}{e}^{\mu} + \overset{i}{e}_{\nu}\, \underset{k}{e}^{\mu}\, (\partial_{\mu}\, \underset{j}{e}^{\nu}).$

Here, $[jk,m]_a$ and $\left\{ {i \atop jk} \right\}_a$ are the christoffel symbols of the first and second kind, respectively defined by $a_{\lambda\mu}$.

Theorem (2.3). The nonholonomic components of the christoffel symbols of the second kind may be expressed as

(2.4) $\qquad \left\{ {i \atop jk} \right\}_a = -\underset{j}{e}^{\nu}\, \underset{k}{e}^{\mu} \; {}_{\mu}\overset{i}{e}_{\nu}.$

Where ∇_{μ} is the symbol of the covariant derivative with respect to $\left\{ {\nu \atop \lambda\mu} \right\}_a$.

In this section, we consider the inverse of the obtained previous results, reconstruct and in vestigate the relationships between the holonomic and nonholonomic components of the christoffel symbols.

3. HOLONOMIC AND NONHOLONOMIC COMPONENTS OF CHRISTOFFEL SYMBOLS IN V_n.

Let $a_{\lambda\mu}$ and a_{ij} are holonomic and nonholonomic components of the tensor and take a coordinate system y^i for which we have at a point p of V_n

(3.1) a $\qquad \frac{\partial y^i}{\partial x^{\lambda}}=\overset{i}{e}_{\lambda}, \quad \frac{\partial x^{\nu}}{\partial y^i}=\underset{i}{e}^{\nu}.$

We have

Theorem (3.1). The holonomic components of the christoffel symbols, as follows ;

(3.2) a $\qquad [\lambda\mu,\omega]_a=[jk,m]_a\, \overset{j}{e}_{\lambda}\, \overset{k}{e}_{\mu}\, \overset{m}{e}_{\omega} + a_{jk}(\partial_{\mu}\, \overset{j}{e}_{\lambda})\, \overset{k}{e}_{\omega}$

(3.2) b $\qquad \left\{ {\alpha \atop \beta\gamma} \right\}_a = \left\{ {i \atop jk} \right\}_a \underset{i}{e}^{\alpha}\, \overset{j}{e}_{\beta}\, \overset{k}{e}_{\gamma} - (\partial_{\gamma}\, \underset{j}{e}^{\alpha})\, \overset{j}{e}_{\beta}$

$\qquad = \left\{ {i \atop jk} \right\}_a \underset{i}{e}^{\alpha}\, \overset{j}{e}_{\beta}\, \overset{k}{e} + (\partial_{\gamma}\, \overset{j}{e}_{\beta})\, \underset{j}{e}^{\alpha}.$

Proof. From (1.4) b,

(3.3) $\qquad a_{\lambda\mu}=a_{jk}\, \overset{j}{e}_{\lambda}\, \overset{k}{e}_{\mu}.$

Differentiating with respec to the coordinate system x^{ω} of V_n. We have

(3.4) $\qquad \partial_{\omega}(a_{\lambda\mu})=\partial_m(a_{jk})\, \overset{j}{e}_{\lambda}\, \overset{k}{e}_{\mu}\, \overset{m}{e}_{\omega} + a_{jk}(\partial_{\omega}\, \overset{j}{e}_{\lambda})\, \overset{k}{e}_{\mu} + a_{jk}\, \overset{j}{e}_{\lambda}(\partial_{\omega}\, \overset{k}{e}_{\mu})$

The following equation (3.5) a is obtain from (3.4) by interchanging ω and μ, m and k throughout, (3.5)b by interchaging ω and λ, m and j ;

(3.5) a $\qquad \partial_{\mu}(a_{\lambda\omega})=\partial_k(a_{jm})\, \overset{j}{e}_{\lambda}\, \overset{m}{e}_{\omega}\, \overset{k}{e}_{\mu} + a_{jm}(\partial_{\mu}\, \overset{j}{e}_{\lambda})\, \overset{m}{e}_{\omega} + a_{jm}\overset{j}{e}_{\lambda}\, (\partial_{\mu}\, \overset{m}{e}_{\omega})$

(3.5) b
$$\partial_\lambda(a_{\mu\omega}) = \partial_j (a_{km})\overset{k}{e}_\mu \overset{m}{e}_\omega \overset{j}{e}_\lambda + a_{km}(\partial_\lambda \overset{k}{e}_\mu)\overset{m}{e}_\omega + a_{km}\overset{k}{e}_\mu (\partial_\lambda \overset{m}{e}_\omega)$$

The sum of (3.5)a and (3.5)b substract (3.4) and divide by 2, and by means of (3.3), we have the first relation (3.2) a as in following ways;

(3.6)
$$[\lambda\mu,\omega]_a = [jk,m]_a \overset{j}{e}_\lambda \overset{k}{e}_\mu \overset{m}{e}_\omega + a_{jk}(\partial_\mu \overset{j}{e}_\lambda)\overset{k}{e}_\omega = [jk,m]_a \overset{j}{e}_\lambda \overset{k}{e}_\mu \overset{m}{e}_\omega + a_{jk}(\partial_p \overset{j}{e}_\lambda)\overset{p}{e}_\mu \overset{k}{e}_\omega.$$

Multiplying both sides of (2.3) a by $\overset{j}{e}_\alpha \overset{k}{e}_\beta \overset{m}{e}_\gamma$, according to (1.4)a and

(3.7) $\underset{k}{e}^\theta = \underset{k}{e}^\lambda \delta^\theta_\lambda.$

We have the same results as (3.6).

The second relation (3.2) b may be obtain by multiplying $\underset{i}{e}^\alpha \overset{j}{e}_\beta \overset{k}{e}_\gamma$ to both sides of (2.3) b and using (1.3) and (2.2), (3.7)

$$\left\{{i \atop jk}\right\}_a \underset{i}{e}^\alpha \overset{j}{e}_\beta \overset{k}{e}_\gamma = \left\{{\nu \atop \lambda\mu}\right\}_a \overset{i}{e}_\nu \underset{j}{e}^\lambda \underset{k}{e}^\mu \underset{i}{e}^\alpha \overset{j}{e}_\beta \overset{k}{e}_\gamma + \overset{i}{e}_\nu \underset{k}{e}^\mu \underset{i}{e}^\alpha \overset{j}{e}_\beta \overset{k}{e}_\gamma (\partial_\mu \underset{j}{e}^\nu)$$
$$= \left\{{\alpha \atop \beta\gamma}\right\}_a + (\partial_\gamma \underset{j}{e}^\alpha)\overset{i}{e}_\beta$$
$$= \left\{{\alpha \atop \beta\gamma}\right\}_a - \delta^\alpha_\beta (\partial\gamma \overset{j}{e}_\alpha)\underset{j}{e}^\alpha$$
$$= \left\{{\alpha \atop \beta\gamma}\right\}_a - (\partial_\gamma \overset{j}{e}_\beta)\underset{j}{e}^\alpha.$$

Theorem (3.2). The holonomic components of the christoffel symbols of the second kind may be expressed as

(3.8)
$$\left\{{\alpha \atop \beta\gamma}\right\}_a = -\overset{j}{e}_\beta \overset{k}{e}_\gamma (\nabla_k \underset{j}{e}^\alpha) = \overset{k}{e}_\gamma \underset{j}{e}^\alpha (\nabla_k \overset{j}{e}_\beta)$$

Where ∇_k is the symbol of the covariant derivative with respect to $\left\{{i \atop jk}\right\}_a$

Proof. Using (2.1) a and (3.7), We havv (3.8) from (3.2) b as in the following way;

$$\left\{{\alpha \atop \beta\gamma}\right\}_a = \left\{{i \atop jk}\right\}_a \underset{i}{e}^\alpha \overset{j}{e}_\beta \overset{k}{e}_\gamma - (\partial_\gamma \underset{j}{e}^\alpha)\overset{j}{e}_\beta$$
$$= \left\{{i \atop jk}\right\}_a \underset{i}{e}^\alpha \overset{j}{e}_\beta \overset{k}{e}_\gamma - (\partial_m \underset{j}{e}^\alpha)\overset{j}{e}_\beta \overset{m}{e}_\gamma$$
$$= -\overset{j}{e}_\beta \overset{k}{e}_\gamma \left(\partial_k \underset{j}{e}^\alpha - \left\{{i \atop jk}\right\}_a \underset{i}{e}^\alpha\right)$$
$$= -\overset{j}{e}_\beta \overset{k}{e}_\gamma (\nabla_k \underset{j}{e}^\alpha)$$
$$= \underset{j}{e}^\alpha \overset{k}{e}_\gamma (\nabla_k \overset{j}{e}_\beta).$$

Literatures Cited

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〈국문초록〉

V_n 공간에서 Christoffel symbol의 Non-holonomic components에 관하여 (I)

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본 논문은 앞 논문(현진오·김홍기, 1980)에서 얻어진 결과의 역을 증명함으로써 christoffel symbol의 holonomic과 nonholonomic component 사이의 관계를 더욱더 명확히 하고 이에 대한 효율적이고 새로운 표현방법을 연구했다.