On the Banach sqace c_0

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Banach 空間 c_0에 關하여

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Summary

In this paper, we treat the relation between weakly compactness and compactness in the Banach space c_0,

0. Introduction

Lindenstrauss conjectured that the second dual X^{**} of a Banach space X is injective if and only if X contains a subspace isomorphic to c_0. The author tries to investigate the properties of c_0 and operator on it systematically for the study of the conjecture.

1. The basic properties of c_0

Let l_∞ be the space of all bounded sepuences of real numbers, c the space of convergent sequences and c_0 the space of sequences converging to 0, all of which are equipped with the sup-norm $\|(\xi_i)\|=sup_i|\xi_i|$. We note that these are normed linear spaces under the pointwise addition and multiplication by reals and the sup-norm.

Theorem 1. c_0, c and l_∞ are real Banach spaces. c_0 is a closed subspace of c and c is a closed subspace of l.

For the proof see 2, pp.218—219. Closedness of c_0 in c(or c is in l_∞) is clear since limit point of c_0(or c) induces a Cauchy sequence in c_0(or c).

Theorem 2. c_0 is topologically isomorphic to c.

Proof. For each (ξ_i) in c converging to ξ, define T from c into c_0 by

$$T(\xi_1,\ \xi_2,\cdots)=(\xi,\ \xi_1-\xi,\ \xi_2-\xi,\cdots).$$

Then $\|T\|=\|T^{-1}\|=2$ and so T is the required isomorphism.

Theorem 3. c is Banach isomorphic to $c_0 \oplus R$.

Proof For each $x=(x_i)$ in c let $t=lim_i x_i$. We can put $x=x_0+te$ where x_0 is in c_0 and $e=(1,1,1,\cdots)$. Define $T: c \to c_0 \oplus R$ by $x \to (x_0, t)$, where $c_0 \oplus R$ is a Banach space with the norm $\|(x_0, t)\|=sup_i|x_i+t|$. Then $\|T\|=\|T^{-1}\|=1$ and $\|T(x)\|=\|x\|$. Therfore $c=c_0 \oplus R$.

Theorem4. c_0^* and c^* are isometrically isomorphic to l_1.

Proof. We shall first prove for c_0^* that it is isometrically isomorphic to l_1.

If $y=(\eta_i)\epsilon l_1$ add $\wedge x=\sum\xi_i\eta_i$ for every $x=(\xi_i)\epsilon c_0$, then $\wedge$ is a bounded linear functional on c_0, since $|\wedge x|\leq\sum|\eta_i|=\|y\|_1$ for any $x \epsilon c_0$ with $\|x\|=1$. We claim that $\|\wedge\|=\|y\|_1$. In fact for any

$n \geq 1$, let $\xi_i = sgn\ \eta_i$ for $1 \leq i \leq n$ and $\xi_i = 0$ for $i > n$. Then $x = (\xi_i)$ is in c_o, $\|x\| = 1$ and so $\wedge x = \sum |\eta_i| \leq \|\wedge\|$ for every n, that is $\|y\|_1 = \sum_1^\infty |\eta_i|$. Therefore $\|\wedge\| = \|y\|_1$.

Next, let's show every $\wedge \in (c_o)^*$ is obtained in this wey. Let $e_i = (0\ 0, \ldots, 1, 0, \cdots,)$ where 1 is in the i-th place and there are zeros in other places. We know that $\{e_1, e_2, \cdots\}$ is a basis of c_o. Let $\wedge \in c_o^*$ and $\wedge(e_i) = \eta_i$. By linearity and continuity of $\wedge$, $\wedge(x) = \sum \xi_i \eta_i$ for any $x = (\xi_i) \in c_o$ We claim $(\eta_i) \in l_1$. For any $n \geq 1$, let $\xi_i = sgn\ \eta_i$ for $1 \leq i \leq n$, and $\xi_i = 0$ if $i > n$. Then $x = (\xi_i) \in c_o$, $\|x\| = 1$ and so $|\wedge(x)| = \sum_1^n |\eta_i| \leq \|\wedge\| < \infty$.
Thus $\sum_1^\infty |\eta_i| \leq \|\wedge\|$, *i.e.* $y = (\eta_i) \in l_1$.

Define T from c_0^* to l_1 by $\wedge \mapsto y$, where $y = (\eta_i)$, $\wedge(x) = \sum \xi_i \eta_i$ for $x = (\xi_i) \in c_o$. Then T is obviously one to one, onto and linear. Futhermore T is norm-preserving.

By the similar method c^* is isometrically isomorphic to l_1.

2. Weakly compact subsets in c_o

Lemma 1. Let x_n and x be in c_o. $x_n = (a_i^n)$ converges weakly to $x = (a_i)$ if and only if $\{x_n\}$ is bounded and $lim_n\ a_i^n = a_i$ for each i.

Proof. c_o is naturally imbedded in l_1^*. If x_n converges weakly to x, then by the Banach-Steinhaus Theorem $\{\|x_n\|\}$ is bounded. Hence $\{x_n\}$ is a bounded sequence. Now since each e_i belongs to l_1, $x_n e_i$ converges to $x e_i$ which gives the fact that $lim_n\ a_i^n = a_i$.

Now assume that $\{x^n\}$ is bounded and $lim_n\ a_i^n = a_i$. Let $z = (b_i) \in l_1$. Since $\sum_1^\infty |b_i| < \infty$, for any $\varepsilon > 0$ there exists N such that $\sum_{N+1}^\infty |b_i| < \varepsilon$. Since for each i $lim_n\ a_i^n = a_i$, for the given $\varepsilon > 0$ there is M such that $n > M$ implies $|a_i^n - a_i| < \varepsilon$, $i = 1, 2, \cdots, N$. Then if $n > M$, $|x_n z - x z| = |\sum a_i^n b_i - \sum a_i b_i| \leq \sum |a_i^n - a_i| |b_i| = \sum^N |a_i^n - a_i| |b_i| + \sum_{N+1}^\infty |a_i^n - a_i| |b_i| \leq \varepsilon \sum^N |b_i| + \|x_n - x\| \sum^\infty |b_i| < \varepsilon(\alpha + \beta)$
since $\{x_n\}$ is bounded from the assumption.

We shall give an example that the condition that $\{x_n\}$ is bounded is essential in the above lemma.

Example 1. Let $x_n = n^2 e_n$, $x = 0$ in c_o, where e_n is the staudard basis, $z = (1/1^2, 1/2^2, 1/3^2, \cdots)$ in l_1. Then $lim_n\ a_i^n = 0$ for each i, but $|x_n z - x z| = 1$. $(x_n = (a_i^n))$.

Theorem 1. Let K be a subset of c_o. Then the following two statements are equivalent.

1) K is relatively weakly compact.

2) K is bounded and the closure of K in the product topology is a compact subset of c_o in the weak topology.

Proof. Note that $c_o \subset R^{N_o}$ and the product topology is the weak topology induced by the set of projections $\subset c_o^*$.

If 1) holds, the closure of K in the weak topology of c_o is also compact in the product topology. Also since a continuous functional on a compact set is bounded, the set $x^* K$ is bounded for any x^* in c_o^* and by the Banach-Steinhaus Therem, K is bounded. Now let x be in the closure of K in the product topology. Then we can choose a sequence $\{x_n\}$ in K such that $\{x_n\}$ converges to x in the product topology. Now Since K is weakly compact, by the Eberleine theorem a subsequence of $\{x_n\}$ converges weakly to some y in c_o. But by lemma 1, $x = y$. Therfore x is in c_o. Now since K is bounded, the closure of K in the product topology is compact in the weak topology.

Suppose 2) holds. Then the closure of K in the product topology is a compact subset of c_o in the weak topology. Note that the closure of K in the weak topology is contained in the closure in the product topology. Since a closed subset of a compact set in a Hausdorff space is also compact, K is relatively compact.

Example 2. Let $K=\{e_i : i=1,2,\cdots\}$, where e_i is the standard basis. Then K is relatively weakly compact, but not relatively compact.

Definition. Let X and Y be Banach spaces. A linear operator T from X to Y is said to be compact (weakly compact) if T maps the closed unit ball of X to a relatively compact(relatively weakly compact) subset of Y.

Lemma 2. Let $\{x_n\}$ be a sequence in l_1. x_n converges weakly to x if and only if x_n converges to x. Moreover, relative compactness and relatively weakly compactness are the same in the space l_1.

Proof. If x_n converges to x, then clearly x_n converges weakly to x since the weak topology is weaker than the original topology.

Suppose that x_n converges weakly to x where $x_n=(a_i^n)$, and $x=(a_i)$. Then for any $\wedge\in l_1^*$, $\wedge(x_n)\rightarrow\wedge x$ as $n\rightarrow\infty$. Note that $l_1^*=l_\infty$, in other words, for any $\wedge\in l_1^*$, there is one and only one $y=(b_i)\in l_\infty$ such that $\wedge x=\sum a_ib_i$, $\|\wedge\|_1=\|y\|$ for any $x=(a_i)\in l_1$. Therefore for any $\wedge\in l_1^*$, $\sum(a_i^n-a_i)b_i\rightarrow 0$ as $n\rightarrow\infty$. Put $b_i=\text{sgn}\,(a_i^n-a_i)$. Then $\|y\|=\|(b_i)\|=1$ and also y is in l_∞. Therefore $\wedge(x_n-x)=\sum|a_i^n-a_i|\rightarrow 0$ as $n\rightarrow\infty$. Since $\|x_n-x\|_1=\sum|a_i^n-a_i|$, the lemma is proved.

Theorem 2. Let T be an operator from c_0 to itself. Then T is compact if and only if T is weakly compact.

Proof. Note that T is compact if and only if T^* is compact on l_1. By lemma 2, T^* is compact if and only if T^* is weakly compact. Thus the theorem is proved.

Lemma 3. Suppose E is a convex subset of c_0. Then the weak closure of E is equal to its original closure.

Proof. Let $\bar{E}_w$ be the weak closure of E. $\bar{E}_w$ is weakly closed, hence originally closed, so that $\bar{E}\subset\bar{E}_w$. To prove the rest, choose $x_0\in c_0$, $x_0\notin\bar{E}$. Then there exists $\wedge\in c_0^*$ and $r\in R$ such that for every $x\in\bar{E}$,

$$Re\wedge x_0<r<Re\wedge x.$$

The set $\{x : Re\wedge x<r\}$ is therefore a weak neighborhood of x_0 that dose not intersect E. Thus x_0 is not in $\bar{E}_w$.

Theorem 3. Let $\{x_n\}$ be a sequence in c_0 that converges weakly to a $x\in c_0$. Then there is a sequence Then there is a sequence $\{y_i\}$ in c_0 such that

a) each y_i is a convex combination of finitely many x_n,

b. $y_i\rightarrow x$ with respect to the sup-norm.

Proof. Let P be the convex hull of the set of all x_n, and let $\bar{P}_w$ be the weak closure of P. Then $x\in P_w$. By lemma 3, x is also in the original closure $\bar{P}$ of P_p. It follows that there is a sequence $\{y_i\}$ in P that converges originally to x.

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l_1에서 약위상의 개념이 원위상과 일치함을 이용, 정의역과 공변역이 모두 c_0인 선형함수가 compact가 되기 위한 필요충분조건이 weakly compact임을 밝히고, c_0에서 약위상적 수렴은 sup-norm으로 주어진 거리공간 c_0에서의 수렴과 어떤 관계가 있는가를 밝혔다.