On Nearness Structures of T_1 Topological Spaces

Song Seok-zun

T_1 位相空間의 Nearness 構造에 관하여

宋 錫 準

I. Introduction and Preliminaries

The concepts of nearness spaces were first introduced by Herrlich in [7]. It has been proved to be a useful tool in the classification of extensions of topological spaces; see for examples [3], [5] and [8]. Bently [3], Herrlich [8], Reed [10] and others have used nearness to classify the principal T_1 extensions of a T_1 spaces. In [5], Dean generalize Ree's result to classify the principal T_0 extensions of a T_0 spaces.

In this paper, we isolate a wide class of nearness structures, called the nearness structures with generating collections, that are induced by T_1 extensions of a particular type. These we shall call T_1 extensions generated by co-countable open sets. The set of nearness structures with generating collections compatible with a symmetric topological space is a complete lattice. We show that in any serious investigation of the lattice of nearness structures compatible with a T_1 topological space, these structures with generating collections will play a special role. This paper is concluded with applications; The category of nearness spaces with generating collections and bijective nearness preserving maps is bicoreflective in the category of nearness spaces and bijective nearness preserving maps.

Let X be set and $\xi \subset P^2X$ and consider the following axioms;

(N1) If $\mathbf{B} \in \xi$ and $\mathbf{A}$ corefines $\mathbf{B}$ (i.e. for each $A \in \mathbf{A}$ there exists $B \in \mathbf{B}$ such that $B \subset A$) then $\mathbf{A} \in \xi$

(N2) If $\cap \mathbf{A} \neq \phi$ then $\mathbf{A} \in \xi$

(N3) $\phi \neq \xi \neq P^2X$

(N4) If $\mathbf{A} \cup \mathbf{B} = \{A \cup B : A \in \mathbf{A}\ B \in \mathbf{B}\} \in \xi$, then $\mathbf{A} \in \xi$ or $\mathbf{B} \in \xi$

(N5) If $cl_\xi \mathbf{A} \in \xi$, then $\mathbf{A} \in \xi$. ($cl_\xi A = \{x \in X : \{\{x\}, A\} \in \xi\}$ and $cl_\xi \mathbf{A} = \{cl_\xi A : A \in \mathbf{A}\}$.)

Definition 1.1. [8] (X, ξ) is called *a nearness space or N-space* if and only if ξ satisfies (N1)-(N5).

This space was introduced by H. Herrlich [8].

Definition 1.2. If (X, ξ) and (Y, η) are N-spaces; then a function $f:(X, \xi) \to (Y, \eta)$ is called *a nearness preserving map* if and only if $\mathbf{A} \in \xi$ implies that $f(\mathbf{A}) \in \eta$.

. **Definition 1.3.** Nearness ξ on X is *compatible with a topology* t on X if and only if $cl_\xi(A) = \overline{A}$

for all $A \subset X$. (i.e. the given topology is equal to the topology induced by the nearness structure ξ.)

Throughout this paper, for the other definitions, we use the definitions of Bang [1] and [2] (or the definitions of Herrlich [8] and [7].)

II. Nearness Structures with Generating Collections.

Definition 2.1. Let (X, t) be a T_1 topological space. Let $A, D \subset X$ and $\mathbf{A}, \mathbf{D} \subset PX$. Let I be a set and $\mathbf{D}_i \subset PX$ for each $i \in I$. Define;

(1) $\xi(D) = \{\mathbf{A} \subset PX: \cap\overline{\mathbf{A}} = \cap\{\overline{A}: A \in \mathbf{A}\} \neq \phi\} \cup \{\mathbf{A} \subset PX: \overline{A} \cap D$ has uncountable elements of X for each $A \in \mathbf{A}\}$

(2) $\xi(\mathbf{D}) = \{\mathbf{A} \subset PX: \cap\overline{\mathbf{A}} \neq \phi\} \cup \{\mathbf{A} \subset PX:$ for each $A \in \mathbf{A}$, there exists $D \in \mathbf{D}$ such that $\overline{A} \cap D$ has uncountable elements of X$\}$.

(3) $\xi[\mathbf{D}] = \{\mathbf{A} \subset PX: \cap\overline{\mathbf{A}} \neq \phi\} \cup \{\mathbf{A} \subset PX:$ there exist $D \in \mathbf{D}$ such that $\overline{A} \cap D$ has uncountable elements of X for each $A \in \mathbf{A}\}$

(4) $\xi(\{\mathbf{D}_i : i \in I\}) = \{\mathbf{A} \subset PX: \cap\overline{\mathbf{A}} \neq \phi\} \cup \{\mathbf{A} \subset PX:$ there exists $i \in I$ such that for each $A \in \mathbf{A}$ there exists $D \in \mathbf{D}_i$ such that $\overline{A} \cap D$ has uncountable elements of X$\}$.

Remark 2.2. We can rewrite each of the notations in Definition 2.1 as follows;

(1) $\xi(D) = \xi(\{\mathbf{D}_1\}: I = \{1\}$ and $\mathbf{D}_1 = \{D\}\})$
(2) $\xi(\mathbf{D}) = \xi(\{\mathbf{D}_1\}: I = \{1\}$ and $\mathbf{D}_1 = \mathbf{D}\})$
(3) $\xi[\mathbf{D}] = \xi(\{\mathbf{D}_C\}: I = \mathbf{D}$ and $\mathbf{D}_C = D\})$

And we have that $\xi[\mathbf{D}] = \cup\{\xi(D): D \in \mathbf{D}\}$ and $\xi(\{\mathbf{D}_i : i \in I\}) = \cup\{\xi(\mathbf{D}_i): i \in I\}$.

Theorem 2.3. Let (X,t) be a T_1 topological space. Let I be a set and $\mathbf{D}_i \subset PX$ for each $i \in I$. Then $\xi = \xi(\{\mathbf{D}_i : i \in I\})$ is a nearness structure compatible with T_1 topology t. And hence $\xi(D)$, $\xi(\mathbf{D})$ and $\xi[\mathbf{D}]$ are also nearness structures compatible with t.

Proof. (N1) If $\mathbf{B} \in \xi$ and $\mathbf{A}$ corefines $\mathbf{B}$, then case 1) if $\cap\overline{\mathbf{B}} \neq \phi$ then $\cap\overline{\mathbf{A}} \neq \phi$ by definition of corefineness, and hence $\mathbf{A} \in \xi$, case 2) if there exists $i \in I$ such that for each $B \in \mathbf{B}$ there exists $D \in \mathbf{D}_i$ such that $\overline{B} \cap D$ has uncountable elements of X, then for each $A \in \mathbf{A}$, there exists a $B \in \mathbf{B}$ such that $B \subset A$ because $\mathbf{A}$ corefines $\mathbf{B}$. Hence there exists $i \in I$ such that for each $A \in \mathbf{A}$, there exist above given $D \in \mathbf{D}_i$ and $B \in \mathbf{B}$ such that $\overline{A} \cap D \supset \overline{B} \cap D$. Therefore $\overline{A} \cap D$ has also uncountable elements of X, i.e. $\mathbf{A} \in \xi$.

(N2), (N3) are trivially satisfied.

(N4) Suppose $\mathbf{A} \notin \xi$ and $\mathbf{B} \notin \xi$. Then $\cap\overline{\mathbf{A} \vee \mathbf{B}} = \phi$ And for each $i \in I$, there exists $A \in \mathbf{A}$ and $B \in \mathbf{B}$ such that $\overline{A} \cap D$ and $\overline{B} \cap D$ has countable elements of X for all $D \in \mathbf{D}_i$. Hence $(\overline{A \cup B}) \cap D = (\overline{A} \cap D) \cup (\overline{B} \cap D)$ has countable elements of X for all $D \in \mathbf{D}_i$. Thus $\mathbf{A} \vee \mathbf{B} \notin \xi$.

(N5) For any $A \subset X$ and $x \in \overline{A}$, we have $\{x\} \subset \overline{\{x\}} \cap \overline{A}$. Then $\{\{x\}, A\} \in \xi$ and $x \in cl_\xi A$. Hence $\overline{A} \subset cl_\xi A$. Conversely, if $x \in cl_\xi A$, then $\{\{x\}, A\} \in \xi$. Since $\overline{\{x\}} \cap D$ is countable for each $D \subset X$, (because (X, t) is T_1 space), it follows that $\overline{\{x\}} \cap \overline{A} \neq \phi$. Hence $x \in \overline{A}$ and $cl_\xi A = \overline{A}$ for all $A \subset X$. Suppose that $cl_\xi \mathbf{A} \in \xi$. If $\cap\overline{cl_\xi \mathbf{A}} \neq \phi$, then since $cl_\xi \mathbf{A} = \overline{\mathbf{A}}$, $\cap\overline{\mathbf{A}} \neq \phi$. Hence $\mathbf{A} \in \xi$. And if there exists $i \in I$ such that for each $cl_\xi A \in cl_\xi \mathbf{A}$, there exists $D \in \mathbf{D}_i$ such that $\overline{cl_\xi A} \cap D$ has uncountable elements of X, then there exists the $i \in I$ such that for each $A \in \mathbf{A}$ there exists $D \in \mathbf{D}_i$ such that $\overline{A} \cap D = \overline{cl_\xi A} \cap D$ has uncountable elements of X. Hence $\mathbf{A} \in \xi$. Moreover, the fact that ξ is compatible with t is shown in the proof of (N5).

Definition 2.4. The given ξ in Theorem 2.3 is called compatible (with t) nearness structure on X with generating collection $\{\mathbf{D}_i : i \in I\}$.

Theorem 2.5. For any T_1 topological space (X, t), the set $S = \{\xi : \xi$ is a compatible nearness structure on X with generating collection$\}$ is a complete lattice with respect to inclusion. Especially

(1) The discrete nearness $\xi(\phi)=\{\mathcal{A}\subset PX : \cap\bar{\mathcal{A}}\neq\phi\}$ is the smallest compatible nearness structure on X with generating collection.

(2) The indiscrete nearness $\xi(X)=\xi(\phi)\cup\{\mathcal{A}\subset PX : \bar{A}$ has uncountable elements of X for each $A\in\mathcal{A}\}$ is the largest compatible nearness structure on on X with generating collection.

(3) If $\Omega=\{\xi_i : i\in I, \xi_i$ is a compatible nearness structures on X with generating collection$\}\subset S$, $\inf\Omega=\bigcap_{i\in I}\xi_i$ and $\sup\Omega=\bigcup_{i\in I}\xi_i$.

Proof. The proof is evident.

Definition 2.6. Let (X,t) be a T_1 topological space. Let D, $E\subset X$, and $\mathcal{D}\subset PX$. Define

(1) $A(D)=\{A\subset X : \bar{A}\cap D$ has uncountable elements of X $\}$

(2) $A(\mathcal{D})=\{A\subset X$: there exists $D\in\mathcal{D}$ such that $\bar{A}\cap D$ has uncountable elements of X $\}$

(3) $A_p=\{A\subset X : p\in\bar{A}\}$

(4) $D<E$ if U is open and $E-U$ is countable then $D-U$ is countable

(5) $D\sim E$ provided $D<E$ and $E<D$.

Proposition 2.7. Let (X,t) be a T_1 topological space. Then

(1) $A(D)$, $A(\mathcal{D})$ and A_p are stacks.

(2) If X and D have uncountable elements, $A(D)$ and A_p are grills, but not filters in general.

(3) $\xi(D)$ is concrete.

Proof. (1) Since $A(D)\subset$ stack $A(D)$, let $B\in$ stack $A(D)$. Then there exists $A\in A(D)$ such that $A\subset B$. Hence $\bar{A}\cap D\subset\bar{B}\cap D$ and $\bar{B}\cap D$ has uncountable elements of X. Therefore $B\in A(D)$. Hence $A(D)=$ stack $A(D)$. For $A(\mathcal{D})$ and A_p, the proofs are similar.

(2) Since any finite subset $F\subset X$ is not contained in $A(D)$, $A(D)\neq PX$. And $A(D)\neq\phi$ since $X\in A(D)$. Now, if $A\cup B\in A(D)$, then $\overline{A\cup B}\cap D=(\bar{A}\cap D)\cup(\bar{B}\cap D)$ has uncountable element of X. Hence $\bar{A}\cap D$ or $\bar{B}\cap D$ has uncount ole elements of X. That is, $A\in A(D)$ or $B\in A(D)$. Conversely, if $A\in A(D)$ or $B\in A(D)$, it is trivial that $A\cup B\in A(D)$. Hence $A(D)$ is a grill. Similarly A_p is a grill. Next, consider the real space (X, t) with Euclidean topology t. Let A be positive rationals and B be negative rationals and D be irrationals. Then $A(D)$ is not a filter, since $A, B\in A(D)$ but $A\cap B=\phi\notin A(D)$. For A_p, Consider $A=(-\infty, p)$, $B=(p,\infty)$ in (X, t). Then we have that A_p is not a filter.

(3) The clusters are $A_p=\{A\subset X : p\in\bar{A}\}$ for $p\in X$ and $A(D)$. If $\mathcal{D}\in\xi(D)$, then $\cap\bar{\mathcal{D}}\neq\phi$ or $\bar{A}\cap D$ has uncountable elements of X for each $A\in\mathcal{D}$ In case $\cap\bar{\mathcal{D}}\neq\phi$, there exists some $p\in\cap\bar{\mathcal{D}}$ and we have $\mathcal{D}\subset A_p$ In other case, $\mathcal{D}\subset A(D)$. Hence $\xi(D)$ is concrete.

Proposition 2.8. For T_1 topological space (X, t), we have

(1) $D<E$ if and only if $A(D)\subset A(E)$.

(2) $A(D)\subset A(E)$ implies $\xi(D)\subset\xi(E)$.

(3) If $\xi(D)\subset\xi(E)$ and $\cap\overline{A(D)}=\phi$ then $A(D)\subset A(E)$.

Proof. (1) Suppose $D<E$ and let $A\in A(D)$. Suppose $A\notin A(E)$. Then $\bar{A}\cap E$ has countable elements of X. Let $U=X-\bar{A}$. Then $E-U=E\cap U^c=E\cap\bar{A}$ has countable elements of X. Since $D<E$, it follows that $D-U$ is countable but this is contradict to $A\in A(D)$. Hence $A(D)\subset A(E)$. Conversely, suppose $A(D)\subset A(E)$ and let $U\in t$ with $E-U$ countable. Suppose $D-U$ has uncountable elements of X. Let $A=X-U$. Then $\bar{A}\cap D=A\cap D=(X-U)\cap D=(X\cap U^c)\cap D=U^c\cap D=D-U$ has uncountable elements of X. Hence $A\in A(D)\subset A(E)$. But $E\cap\bar{A}=E\cap A=E\cap U^c=E-U$ has countable elements of X. This is contradict. Therefore $D<E$.

(2) If $\mathcal{D}\in\xi(D)$, then $\cap\bar{\mathcal{D}}\neq\phi$ or $\bar{A}\cap D$ has uncountable elements of X for each $A\in\mathcal{D}$ If $\cap\bar{\mathcal{D}}\neq\phi$, then $\mathcal{D}\in\xi(E)$. In the other case, $\mathcal{D}\subset A(D)$. Hence $\mathcal{D}\subset A(E)$ by assumption. Hence $\bar{A}\cap E$ has uncountable elements of X for each $A\in\mathcal{D}$. Hence $\mathcal{D}\subset\xi(E)$.

(3) For any $A\in A(D)$, $\bar{A}\cap D$ has uncountable elements of X. Then $A(D)\in\xi(D)\subset\xi(E)$. Since

$\cap\overline{A(D)}=\phi$, $\bar{A}\cap E$ has uncountable elements of X for each $A\in\mathbf{A}(D)$. Therefore $A\in\mathbf{A}(E)$ and $\mathbf{A}(D)\subset\mathbf{A}(E)$.

Definition 2.9. Let (X, t) be a T_1 topological space. Let $\mathbf{C}$, $\mathbf{D}\subset PX$, and $\mathbf{C}_i\subset PX$ for each $i\in I$ and $\mathbf{D}_j\subset PX$ for each $j\in J$. Define

(1) $\mathbf{C}$ is called concrete provided C, $D\in\mathbf{C}$ and $C<D$ implies $C\sim D$.

(2) $\mathbf{C}<\mathbf{D}$ if for each $C\in\mathbf{C}$, there exists $\mathbf{D}_C\subset\mathbf{D}$ such that if U is open and D−U has countable elements of X for $D\in\mathbf{D}_C$ then C−U has countable elements of X.

(3) $\mathbf{C}\sim\mathbf{D}$ provided $\mathbf{C}<\mathbf{D}$ and $\mathbf{D}<\mathbf{C}$

(4) $\{\mathbf{C}_i:i\in I\}<\{\mathbf{D}_j:j\in J\}$ if $i\in I$ and $A\subset\mathbf{A}(\mathbf{C}_i)$ then there exists $j\in J$ with $A\subset\mathbf{A}(\mathbf{D}_j)$.

(5) $\{\mathbf{C}_i:i\in I\}\sim\{\mathbf{D}_j:j\in J\}$ if $\{\mathbf{C}_i:i\in I\}<\{\mathbf{D}_j:j\in J\}$ and $\{\mathbf{D}_j:j\in J\}<\{\mathbf{C}_i:i\in I\}$.

(6) $\{\mathbf{C}_i:i\in I\}$ is called concrete provided $i\in I$, $j\in J$ and $\mathbf{C}_i<\mathbf{C}_j$ implies $\mathbf{C}_i\sim\mathbf{C}_j$.

Proposition 2.10. For T_1 topological space (X, t), we have

(1) $\mathbf{C}<\mathbf{D}$ if and only if $\mathbf{A}(\mathbf{C})\subset\mathbf{A}(\mathbf{D})$.

(2) $\mathbf{A}(\mathbf{C})\subset\mathbf{A}(\mathbf{D})$ implies $\xi(\mathbf{C})\subset\xi(\mathbf{D})$.

(3) If $\xi(\mathbf{C})\subset\xi(\mathbf{D})$ and $\cap\overline{\mathbf{A}(\mathbf{C})}=\phi$, then $\mathbf{A}(\mathbf{C})\subset\mathbf{A}(\mathbf{D})$.

Proof. (1) Suppose $\mathbf{C}<\mathbf{D}$. Let $A\in\mathbf{A}(\mathbf{C})$. Then there exists $C\in\mathbf{C}$ such that $\bar{A}\cap C$ has uncountable elements of X. Suppose $\bar{A}\cap D$ has countable elements of X for each $D\in\mathbf{D}_C$. Then $U=X-\bar{A}$ is open and $D-U=D\cap U^c=D\cap\bar{A}$ has countable elements of X for each $D\in\mathbf{D}_C$. Hence C−U has countable elements of X. But $C-U=C\cap U^c=C\cap\bar{A}$ has uncountable elements of X. This is a contradiction. Therefore $\mathbf{A}(\mathbf{C})\subset\mathbf{A}(\mathbf{D})$. Conversely, suppose $\mathbf{A}(\mathbf{C})\subset\mathbf{A}(\mathbf{D})$. Let $C\in\mathbf{C}$. Suppose C has uncountable elements of X. (If C has countable elements, then it is trivial.) Put $\mathbf{A}(C)=\{A\subset X:\ \bar{A}\cap C$ has uncountable elements of X$\}$ and $\mathbf{D}_C=\{D:D\in\mathbf{D}$, there exists $A\in\mathbf{A}(C)$ with $\bar{A}\cap D$ has uncountable elements of X$\}$. Let U be open set such that D−U has countable elements for each $D\in\mathbf{D}_C$. Suppose C−U has uncountable elements. Let $A=X-U$. Then $A\in\mathbf{A}(C)$ since $\bar{A}\cap C=A\cap C=U^c\cap C=C-U$ has uncountable elements. And since $\mathbf{A}(\mathbf{C})\subset\mathbf{A}(\mathbf{D})$, there exists $D\in\mathbf{D}_C$ with $\bar{A}\cap D$ uncountable. Then $\bar{A}\cap D=(X-U)\cap D=U^c\cap D=D-U$ has uncountable elements. But D−U has countable elements, we have a contradiction. Therefore $\mathbf{C}<\mathbf{D}$.

(2) Since $\xi(\mathbf{C})=\xi(\phi)\cup\{\mathbf{A}\subset PX:\mathbf{A}\subset\mathbf{A}(\mathbf{C})$, we have that $\mathbf{A}(\mathbf{C})\subset\mathbf{A}(\mathbf{D})$ implies $\xi(\mathbf{C})\subset\xi(\mathbf{D})$.

(3) If $A\in\mathbf{A}(\mathbf{C})$, there exists $C\in\mathbf{C}$ such that $\bar{A}\cap C$ has uncountable elements of X. Then $\mathbf{A}(\mathbf{C})\in\xi(\mathbf{C})\subset\xi(\mathbf{D})$. Since $\cap\overline{\mathbf{A}(\mathbf{C})}=\phi$, for each $A\in\mathbf{A}(\mathbf{C})$, there exists $D\in\mathbf{D}$ such that $\bar{A}\cap D$ has uncountable elements of X. Therefore $A\in\mathbf{A}(\mathbf{D})$ and $\mathbf{A}(\mathbf{C})\subset\mathbf{A}(\mathbf{D})$.

Proposition 2.11. Let (X, t) be T_1 topological space. Then

(1) $\xi(\mathbf{D})$ is concrete nearness structure for $\mathbf{D}\subset PX$

(2) If $\mathbf{D}$ is concrete collection in the sense of definition 2.9(1), then $\xi[\mathbf{D}]$ is also concrete nearness structure.

(3) $\xi[\mathbf{D}]$ is concrete nearness structure if and only if there exists a concrete collection $\mathbf{C}$ such that $\xi[\mathbf{D}]=\xi[\mathbf{C}]$.

(4) If $\{\mathbf{D}_i:i\in I\}$ is concrete collection then $\xi(\{\mathbf{D}_i:i\in I\})$ is concrete nearness structure.

(5) $\xi(\{\mathbf{D}_j:j\in J\})$ is concrete nearness structure if and only if there exsits a concrete collection $\{\mathbf{C}_i:i\in I\}$ such that $\xi(\{\mathbf{D}_j:j\in J\})=\xi(\{\mathbf{C}_i:i\in I\})$.

Proof. (1) $\xi(\mathbf{D})=\{\mathbf{A}\subset PX:\cap\bar{\mathbf{A}}\neq\phi\}\cup\{\mathbf{A}\subset PX$: for each $A\in\mathbf{A}$ there exists $D\in\mathbf{D}$ such that $\bar{A}\cap D$ has uncountable elements of X$\}=\xi(\phi)\cup\{\mathbf{A}\subset PX:\mathbf{A}\subset\mathbf{A}(\mathbf{D})\}$. If $\mathbf{D}$ is empty or contains only the sets which have countable elements, then $\xi(\mathbf{D})=\xi(\phi)$ and hence is concrete. If $\mathbf{D}$ contains an uncountable set then the clusters in $\xi(\mathbf{D})$ are of the form $\mathbf{A}_p=\{A\subset X:p\in\bar{A}\}$ for $p\in X$ or $\mathbf{A}(\mathbf{D})$. Hence for each $\mathbf{A}\in\xi(\mathbf{D})$, if $\cap\bar{\mathbf{A}}\neq\phi$ then

$A\subset A_p$ for some $p\in\cap\bar{A}$, and if $A\subset A(D)$, A is contained in a cluster $A(D)$. Hence $\xi(D)$ is concrete.

(2) Suppose **D** is concrete. If **D** is empty or consists only of the sets which have countable elements then $\xi[D]=\xi(\phi)$ and thus is concrete. If **D** has the set which have uncountable elements then the clusters in $\xi[D]$ are of the form A_p for $p\in X$ or $A(D)$, where $D\in\mathbf{D}$ and D has uncountable elements of X. Each $A\in\xi[D]$ is contained in one of these clusters, and thus $\xi[D]$ is concrete.

(3) Suppose $\xi[D]$ is concrete. Let $C=\{D\in\mathbf{D}: A(D)$ is a cluster in $\xi[D]\}$. If $C=\phi$ then we are through. Suppose $D\in C$ and $E\in C$ with $D<E$. Then by Proposition 2.8 (1), $A(D)\in A(E)$. But $A(D)$ is a cluster and hence $A(D)=A(E)$. Thus $D\sim E$ and **C** is concrete. If $A\in\xi[D]$ and $\cap\bar{A}\neq\phi$ then $A\in\xi[C]$. Otherwise, since $\xi[D]$ is concrete, there exists a cluster of the form $A(D)$ with $A\subset A(D)$ and $D\in\mathbf{D}$. Hence $D\in C$ and thus $A\in\xi[C]$. Therefore $\xi[C]=\xi[D]$. Conversely, if there exists a concrete collection such that $\xi[D]=\xi[C]$, then $\xi[D]$ is concrete nearness structure by (2).

(4) Let $\{D_i:i\in I\}$ be a concrete collection. Let $\xi=\xi\{D_i:i\in I\})$. The clusters in ξ are of the form A_p for $p\in X$ or $A(D_i)$ for $i\in I$ and D_i containing at least one uncountable subset of X. For, let D_i contain at least one uncountable subset of X and suppose that $A(D_i)\subset A(D_j)$ for some D_j which containing an uncountable subset of X. Then by Theorem 2.10 (1), $D_i<D_j$. Since the collection $\{D_i:i\in I\}$ is concrete, we have that $D_i\sim D_j$ and hence $A(D_i)=A(D_j)$. Thus $A(D_i)$ is a cluster. Hence $\xi(\{D_i:i\in I\})$ is concrete nearness structure.

(5) If the condition holds then $\xi(\{D_j:j\in J\})$ is concrete nearness by (4). Conversely, suppose that $\xi=\xi(\{D_j:j\in J\})$ is concrete nearness. Let $I=\{i:i\in J$ and $A(D_i)$ is cluster in $\xi\}$. The cluster in ξ are of the form A_p for $p\in X$ or $A(D_i)$ for $i\in I$. Since ξ is concrete nearness, it follows that $\xi(\{D_j:j\in J\})=\xi(\{C_i:i\in I\})$. To see this; let any $A\in\xi(\{D_j:j\in J\})$. Then A is contained in some cluster $A(C_i)$ for $i\in I$. Hence $A\in\xi(\{C_i:i\in I\})$ by definition 2.1 (4). Now it suffice to show that $\{C_i:i\in I\}$ is a concrete collection. Suppose that $C_i<C_j$ for $i, j\in I$. Then $A(C_i)\subset A(C_j)$ by Theorem 2.10 (1) and since $A(C_i)$ is a cluster, we have $A(C_i)=A(C_j)$. Therefore $C_i\sim C_j$ and hence there exists a concrete collection $\{C_i:i\in I\}$ such that $\xi(\{D_j:j\in J\})=\xi(\{C_i:i\in I\})$.

III. T_1-Extensions Generated by Cocountable Open Sets and Applications.

Definition 3.1. *An extension* (e, Y, t) *of* $(X, t(X))$ is a topological space (Y, t) and a dense embedding $e: X\to Y$. (e, Y, t) is called *strict or principal extension of* $(X, t(X))$ if the collection $\{cl_Y(e(A)):A\subset X\}$ is a base for the closed sets in Y.

We will assume that the embeddings $e:X\to Y$ are injections and thus not distinguish between A and $e(A)$ for $A\subset X$.

Definition 3.2. For (Y, t) an extension of X, we define $\mu_y=\{U\cap X:y\in U\in t\}$ for $y\in Y$. μ_y is called the trace filter of y on X.

Definition 3.3. (Y, t) is called *T_1-extension of X generated by cocountable open sets* if for each $y\in Y-X$ there exists $C_y\subset PX$ such that $\mu_y=\{U\in t(X):C-U$ has countable elements of X for each $C\in C_y\}$.

Theorem 3.4. Let (Y, t) be a T_1-extension of X generated by cocountable open sets. Let $y\in Y-X$ and $A\subset X$. Then $y\in cl_Y A$ if and only if there exists $C\in C_y$ such that $cl_X A\cap C$ has uncountable elements of X.

Proof. Suppose $y\in cl_Y A$. Now $\mu_y=\{U\in t(X):$

$C-U$ has countable elements of X for each $C\in C_y$. Suppose $cl_X A\cap C$ has countable elements of X for each $C\in C_y$. Let $U=X-cl_X A$. Then $U\in t(X)$ and $C-U=C\cap U^c=C\cap cl_X A$ has countable elements of X for each $C\in C_y$. Hence $U\in\mu_y$ and thus there exists $V\in t$ with $y\in V$ and $V\cap X=U$. Then $V\cap A=(V\cap X)\cap A=U\cap A\subset U\cap cl_X A=\phi$, which is contradict to $y\in cl_Y A$. Therefore there exists $C\in C_y$ such that $cl_X A\cap C$ has uncountable elements of X. Conversely, if $y\not\in cl_Y A$, then there exists $V\in t$ with $y\in V$ such that $V\cap A=\phi$ Then $A\cap V\cap X=\phi$, and hence $cl_X A\subset(V\cap X)^c$. Since $V\cap X\in\mu_y$, $C-(V\cap X)$ has countable elements of X for each $C\in C_y$. But $C-(V\cap X)=C\cap(V\cap X)^c\supset C\cap cl_X A$; uncountable, which is impossible. Hence $y\in cl_Y A$.

Example 3.5. The reals can be constructed as a T_1-extension of its subspace generated by cocountable open sets.

Proof. Let R be the set of reals and $S=R-\{0\}$. Let t be a topology on R such that $t=\{G\subset R$: either $0\not\in G$ or if $0\in G$ then $R-G$ has countable elements of reals$\}$. Then (R,t) is a T_1-extension of (S,t(S)) using identity embedding. Put $C_0=\{S\}$. $\mu_0=\{U\in t(S):S-U$ has countable elements of S for any $S\in C_0\}$. Hence (R, t) is a T_1-extension of (S,t(S)) generated by cocountable open sets.

Theorem 3.6. Let (X,t) be a T_1 topological space and $Y=X\cup\{y\}$ with $y\not\in X$. If $t(Y)=t\cup\{U\cup\{y\}:U\in t$ and $Y-U$ has countable elements of $Y\}$, then (Y, t(Y)) is an one point extension of X generated by cocountable open sets.

Proof. For any $x\in X$, take $U\in t(X)$ with $x\in U$. The $y\not\in U$. Since (X, t) is T_1 space, $X-\{x\}\in t(X)$. Hence $(X-\{x\})\cup\{y\}\in t(Y)$. Hence (Y, t(Y)) is T_1 space. It suffices to show that (Y, t(Y)) is an one point extension of X generated by cocountable open sets. For the trace filter of $y\in Y-X$ on X, $\mu_y=\{V\cap X:y\in V\in t(Y)\}=\{U\in t(X)$: $Y-U$ has countable elements of $Y\}=\{U\in t(X)$: $X-U$ has countable elements of $X\}$, there exists $C_y=\{X\}\subset PX$ such that $\mu_y=\{U\in t(X)$: $C-U$ has countable elements of X for each $C\in C_y=\{X\}\}$. The proof is completed.

Theorem 3.7. Let (Y,t) be a T_1-extension of X. Set $\xi=\{\mathbf{D}\subset PX:\cap cl_Y\mathbf{D}\neq\phi\}$. Then Y is an extension of X generated by cocountable open sets if and only if ξ is a compatible nearness structure with generating collection.

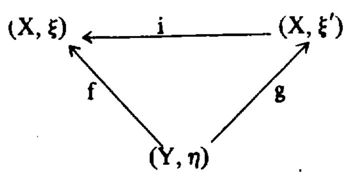
Proof. Suppose Y is a T_1-extension of X generated by cocountable open sets. Then for each $y\in Y-X$, there exists $C_y\subset PX$ such that $\mu_y=\{U\in t(X)$:$C-U$ has countable elements of X for each $C\in C_y\}$. Let $I=Y-X$ and $\eta=\xi(\{C_y: y\in I\})$. It suffices to show that $\xi=\eta$. Let $\mathbf{D}\in\xi$. Then there exists $y\in\cap cl_Y\mathbf{D}$. If $y\in X$ then $\cap cl_X\mathbf{D}\neq\phi$ and $\mathbf{D}\in\eta$. If $y\in Y-X$, then $y\in cl_Y D$ for each $D\in\mathbf{D}$. Then by Theorem 3.4, there exists $C\in C_y$ such that $cl_X D\cap C$ has uncountable elements of X. Hence $\mathbf{D}\in\eta$. On the other hand suppose $\mathbf{D}\in\eta$. If $cl_X\mathbf{D}\neq\phi$ then $\cap cl_Y\mathbf{D}\neq\phi$ and $\mathbf{D}\in\xi$. Otherwise, there exists C_y such that for each $D\in\mathbf{D}$ there exists $C\in C_y$ such that $cl_X D\cap C$ has uncountable elements of X. Then by Theorem 3.4, $y\in cl_Y D$ for each $D\in\mathbf{D}$ and thus $\cap cl_Y\mathbf{D}\neq\phi$ and $\mathbf{D}\in\xi$.

Conversely, suppose ξ is a compatible nearness structure with generating collection. Let $y\in Y-X$ and let $\mu_y=\{U\cap X:y\in U\in t\}$ be the trace filter. Consider $\mathbf{D}_y=\{D\subset X:y\in cl_Y D\}$. Then $\mathbf{D}_y\in\xi$. Hence there exists $\mathbf{C}\subset PX$ such that $\mathbf{D}_y=\{A\subset X$: there exists $C\in\mathbf{C}$ such that $cl_X D\cap C$ has uncountable elements of $X\}$ since $\cap cl_X\mathbf{D}_y=\phi$. It suffices to show that the given trace filter μ_y is equal to $\{U\in t(X)$:$C-U$ has countable elements of X for each $C\in\mathbf{C}\}=\nu_y$. Let $U\in\mu_y$. Then there exists $V\in t$ such that $U=V\cap X$ and $y\in V$. Suppose there exists $C\in\mathbf{C}$ such that $C-U$ has uncountable elements of X. Let $D=C-U$. Since $cl_X D\cap C=cl_X(C-U)\cap C$ has uncountable elements of X, $D\in\mathbf{D}_y$ and thus $y\in cl_Y D$. But this is impossible. Hence $C-U$ has countable elements of X for each $C\in\mathbf{C}$. Thus $\mu_y\subset\nu_y$. On the other hand, let $U\in\nu_y$. Now there exists $S\in t$ such that $S\cap X=U$. Suppose there exists $V\in t$ such that $y\in V$ and $V\cap X\subset U$. Then $y\in S\cup V\in t$ and $(S\cup V)\cap X=U$ and $U\in\mu_y$. Now suppose for each $V\in t$ with $y\in V$, we have that $V\cap X\not\subset U$. Let $x_v\in(V\cap X)-U$. Set $D=\{x_v$: for $y\in V\in t\}$. Then $y\in cl_Y D$ and $D\in\mathbf{D}_y$. Hence there exists $C\in\mathbf{C}$ such that $cl_X D\cap C$ has uncountable elements of X. But $D\cap U=\phi$ implies $cl_X D\cap U=\phi$ Since $U\in\nu_y$ implies $C-U$ has countable elements of X, which is contradict to the fact that $C-U\supset(C\cap cl_X D)-U=(C\cap cl_X D)\cap U^c\supset(C\cap cl_X D)\cap cl_X D=C\cap cl_X D$; uncountable. Hence $U\in\mu_y$ and thus $\mu_y=\nu_y$. Therefore Y is T_1-extension of X generated by cocountable open sets.

Lemma 3.8. Let (X,t) be a T_1 nearness space. Set $\xi'=\cup\{\eta:\eta$ is a compatible nearness structure on X with generating collection and $\eta\subset\xi\}$. Then $\xi'\subset\xi$ and ξ' is a compatible nearness structure with generating collection.

Theorem 3.9. The category of T_1 nearness spaces with generating collections and bijective nearness preserving maps is bicoreflective in the category of T_1 nearness spaces and bijective nearness preserving maps. The coreflection is given by $i:(X,\xi')\to(X,\xi)$.

Proof. Consider the following diagram:



where f is a one-to-one and onto nearness preserving map, (Y,η) is a nearness space with generating collection and $g(y)=f(y)$ for each $y\in Y$. Then g must be unique. Hence it suffices to show that g is a nearness preserving map. Let $\mathbf{A}\in\eta$. If $\cap cl_X(f(\mathbf{A}))\neq\phi$ then $f(\mathbf{A})\in\xi'$. Suppose $\cap cl_X(f(\mathbf{A}))=\phi$. Then $\cap cl_Y\mathbf{A}=\phi$, and since η is compatible nearness structure with generating collection, there exists a generating collection $\mathbf{C}\in PY$ such that for each $A\in\mathbf{A}$, there exists $C\in\mathbf{C}$ such that $cl_Y A\cap C$ has uncountable elements of Y. Since Y is T_1 space, each $A\in\mathbf{A}$ has uncountable elements of Y. Let $\mathbf{D}=\{C$: $C\in\mathbf{C}$ and C has uncountable elements of Y$\}$. Then $\mathbf{D}\in\eta$ and $f(\mathbf{D})\in\xi$. Then $\xi(f(\mathbf{D}))\subset\xi'$. To see this, note that each f(D) has uncountable elements of X for each $D\in\mathbf{D}$ since f is one-to-one. Let $\mathbf{B}\in\xi(f(\mathbf{D}))$. If $\cap cl_X\mathbf{B}\neq\phi$ then $\mathbf{B}\in\xi'$. Suppose $\cap cl_X\mathbf{B}=\phi$. Then, for each $B\in\mathbf{B}$ there exists $D\in\mathbf{D}$ such that $cl_X B\cap f(D)$ has uncountable elements of X. Since X is T_1 space, B has uncountable elements of X and $f^{-1}(B)$ has uncountable elements of Y since f is an onto mapping. Similarly $f^{-1}(cl_X B\cap f(D))$ has uncountable elements of Y, and $f^{-1}(cl_X B\cap f(D))=f^{-1}(cl_X B)\cap f^{-1}(f(D))=f^{-1}(cl_X B)\cap D$ since f is one-to-one. Hence $\{f^{-1}(cl_X B):B\in\mathbf{B}\}\in\eta$, and thus $\{cl_X B:B\in\mathbf{B}\}=\{f(f^{-1}(cl_X B)):B\in\mathbf{B}\}\in\xi$, and thus $\mathbf{B}\in\xi'$ since f is a nearness preserving map. Hence $\xi(f(\mathbf{D}))\subset\xi'$. We claim that $f(\mathbf{A})\in\xi(f(\mathbf{D}))$. For, let $A\in\mathbf{A}$ As previously noted, A has uncountable elements of Y and there exists $C\in\mathbf{D}$ such that $cl_Y A\cap C$ has uncountable elements of Y. Then $f(cl_Y A\cap C)\subset f(cl_Y A)\cap f(C)\subset cl_X(f(A))\cap f(C)$. Now $f(cl_Y A\cap C)$ has uncountable elements of X since f is one-to-one. Thus $cl_X(f(A))\cap f(C)$ has uncountable elements of X for each $A\in\mathbf{A}$ and $f(\mathbf{A})\in\xi(f(\mathbf{D}))$. Now $g(\mathbf{A})=f(\mathbf{A})\in\xi'$. Thus g is a nearness preserving map.

Literature cited

[1] Bang Eun-Sook, 1984. Some properties of separation axioms on nearness structures Cheju National University Journal Vol.18, Natural Sciences, 187-192.

[2] Bang Eun-Sook, 1984. A note on the topological R_0-regular spaces, Cheju National University Journal Vol. 18, Natural Sciences, 193-195.

[3] Bently H.L., 1975. Nearness spaces and extensions of topological spaces, Studies in Topology, Academic Press, New York, 47-66.

[4] Carlson J.W., 1983. Subset generated nearness structures and extensions, Kyungpook Math. J. 23(1) June, 49-61.

[5] Dean A.M., 1983. Nearnesses and T_0-extensions of topological spaces, Canad. Math. Bull 26(4), 430-437.

[6] Hastings M.S., 1982. On heminearness spaces, Proc. Amer. Math. Soc. 86(4), 567-573.

[7] Herrlich H., 1974. A concept of nearness, Gen. Top. Appl. 4, 191-212.

[8] Herrlich H., 1974. Topological structures, Methematical Centre Tracts 52, Amsterdam.

[9] Herrlich H., and Strecker G.E., 1973. Category Theory, Allyn and Bacon, Boston.

[10] Reed E.E., 1978. Nearnesses, proximities, and T_1-compactifications, Trans. Amer. Math. Soc. 236, 193-207.

[11] Thron W.J., 1966. Topological Structures, Holt, Rinehart, and Winston, New York.

國 文 抄 錄

本 論文에서는 生成集合을 갖는 nearness構造를 硏究하였다. 먼저 生成集合을 갖는 nearness構造의 여러가지 性質들을 調査하여 이 nearness構造들의 集合이 完備束이 됨을 보였고, 또 concrete nearness構造가 되는 條件들을 調査하였다.

다음으로 이 nearness構造의 應用으로서 한점 擴張空間을 만들었고, 또 生成集合을 갖는 nearness空間들과 全單射 nearness 보존寫像들의 category가 一般的 nearness空間들과 全單射 nearness 보존寫像들의 category안의 bicoreflective임을 證明하였다.