On the Covariant Differentiation of the Nonholonomic Tensors in Vn

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Vn 공간에서 Nonholonomic Tensor 들의 공변미분에 관하여

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I. Introduction

Let $\underset{i}{e}^{\nu}$ (=1, 2, . . . , n) be a set of n linearly independent vectors in n-dimensional Riemnnian space V_n referred to a real coordinate system x^{ν}. There is a unique reciprocal set of n linearly idenpendent covariant vectors $\overset{i}{e}_{\lambda}$ (j=1, 2, . . . , n) satisfying

$$(1.1)\quad \underset{i}{e}^{\nu}\overset{i}{e}_{\lambda}=\delta^{\nu}_{\lambda}\qquad \underset{j}{e}^{\lambda}\overset{i}{e}_{\lambda}=\delta^{i}_{j}\,(**).$$

Within the vectors $\underset{i}{e}^{\nu}$ and $\overset{j}{e}_{\lambda}$, a nonholonomic fram of V_n defined in the following way.

Definition 1.1. If $T^{\lambda\cdots}_{\nu\cdots}$ are holonomic components of a tensor. Then its nonholonomic components are defined by

$$(1.2)\quad T^{i\cdots}_{j\cdots}\overset{def}{=\!=}T^{\nu\cdots}_{\lambda\cdots}\overset{i}{e}_{\nu}\underset{i}{e}^{\lambda}\cdots.$$

Theorem 1.2. The derivative of $\underset{j}{e}^{\lambda}$ is negative self-adjoint. That is

$$(1.3)\quad \partial_k\,(\overset{j}{e}_{\lambda})\underset{j}{e}^{\mu}-\partial_k\,(\underset{j}{e}^{\mu})\overset{j}{e}_{\lambda}.$$

Theorem 1.3. The holonomic components of the christoffel symbol as follows;

$$(1.4)\text{a}\quad [\lambda\mu,\omega]=[jk,m]\,\overset{i}{e}_{\lambda}\overset{k}{e}_{\mu}\overset{m}{e}_{\nu}+a_{jk}(\partial_{\mu}\overset{i}{e}_{\lambda})\overset{k}{e}_{\mu}$$

$$(1.4)\quad \left\{{\nu \atop \lambda\mu}\right\}=\left\{{i \atop jk}\right\}\underset{i}{e}^{\nu}\overset{j}{e}_{\lambda}\overset{k}{e}_{\mu}-(\partial_{\mu}\underset{j}{e}^{\nu})\overset{i}{e}_{\lambda}$$
$$=\left\{{i \atop jk}\right\}\underset{i}{e}^{\nu}\overset{j}{e}_{\lambda}\overset{k}{e}_{\mu}+(\partial_{\mu}\overset{j}{e}_{\beta})\underset{j}{e}^{\nu}.$$

II. Covariant Differentiation of the Nonholonomic Covariant Tensors in V_n.

We know the derivative of the holonomic covariant and contravariant tensors in V_n.

In this section, reconstruct and prove the derivative of holonomic components which represented by the nonholonomic component with respect to tensors in V_n. Furthermore, we study the derivative of the nonholonomic frame.

Take a coordinate system y^i for which we have at a point p of V_n

$$(2.1)\quad \frac{\partial y^i}{\partial x^{\lambda}}=\overset{j}{e}_{\lambda},\qquad \frac{\partial x^{\nu}}{\partial y^i}=\underset{i}{e}^{\nu}$$

(**) Throughout the present paper, Greek indices take values 1,2, ⋯, n unles explicitly stated otherwise and follow the summation convention, while Roman indices are used for the nonholonomic componts of a tensor and run from 1 to n. Roman in dices also follow the summation convention.

Theorem 2.1. The covariant derivative of the holonomic covariant tensor $T_{\nu\lambda}$ may be expressed in terms of the nonholonomic components.

$$(2.2)\quad T_{\nu\lambda,\mu} = \frac{\partial}{\partial y^k} T_{ij} - T_{\ell j}\left\{{\ell \atop ik}\right\} - T_{i\ell}\left\{{\ell \atop ik}\right\} \overset{i}{e}_\nu \overset{j}{e}_k \overset{k}{e}_\mu$$

Proof. In order to prove (2.2), the derivative of the tensor $T_{\nu\lambda}$ with respect to x^μ interchange to the nonholonomic in the following ways;

From (1.2) and (2.1), we have

$$(2.2)\quad T_{\nu\lambda,\mu} = \frac{\partial}{\partial y^k} T_{ij} - T_{\ell j}\left\{{\ell \atop ik}\right\} - T_{i\ell}\left\{{\ell \atop ik}\right\} \overset{i}{e}_\nu \overset{j}{e}_\lambda \overset{k}{e}_\mu .$$

$$(2.3)\quad \frac{\partial T_{\nu\lambda}}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}(T_{ij} \overset{i}{e}_\nu \overset{j}{e}_\lambda)$$

$$=(\frac{\partial}{\partial y^k} T_{ij}) - \overset{i}{e}_\nu \overset{j}{e}_\lambda \overset{k}{e}_\mu + T_{ij}(\partial_k \overset{i}{e}_\nu) \overset{j}{e}_\lambda \overset{k}{e}_\mu + T_{ij}(\partial_k \overset{j}{e}_\beta)\overset{k}{e}_\mu \overset{i}{e}_\nu .$$

(2.4) can be obtained by making use of (1.2) and (1.4)b

$$(2.4)\quad T_{\omega\lambda}\left\{{\omega \atop \nu\mu}\right\} = T_{\ell j} \overset{\ell}{e}_\omega \overset{j}{e}_\lambda \left\{{\ell \atop ik}\right\} \underset{\ell}{e}^\omega \overset{i}{e}_\nu \overset{k}{e}_\mu + T_{\ell j} \overset{\ell}{e}_\omega \underset{j}{e}^\omega (\partial_\mu \overset{i}{e}_\nu).$$

Similary, we have

$$(2.5)\quad \left\{{T_{\nu\lambda} \atop \lambda\mu}\right\} = T_{i\ell} \overset{i}{e}_\nu \overset{\ell}{e}_\omega \left\{{\ell \atop jk}\right\} \underset{\ell}{e}^\omega \overset{j}{e}_\lambda \overset{k}{e}_\mu + T_{i\ell} \overset{i}{e}_\nu \overset{\ell}{e}_\omega \underset{j}{e}^\omega (\partial_\mu \overset{j}{e}_\lambda).$$

If from (2.3) subtract the sum of these two equations (2.4) and (2.5), and making use of (1.1), we have

$$(2.6)\quad \frac{\partial T_{\nu\lambda}}{\partial x^\mu} - T_{\omega\lambda}\left\{{\omega \atop \nu\mu}\right\} - T_{\nu\omega}\left\{{\omega \atop \lambda\mu}\right\}$$

$$= [\frac{\partial T_{ij}}{\partial y^k} - T_{\ell j}\left\{{\ell \atop ik}\right\} - T_{i\ell}\left\{{\ell \atop jk}\right\}] \overset{i}{e}_\nu \overset{j}{e}_\lambda \overset{k}{e}_\mu$$

$$+ [T_{ij} \overset{k}{e}_\mu \overset{j}{e}_\lambda (\partial_k \overset{i}{e}_\nu) - T_{\ell j}\delta^\ell_j \overset{i}{e}_\lambda (\partial_\mu \overset{i}{e}_\nu)]$$

$$+ [T_{ij} \overset{k}{e}_\mu \overset{i}{e}_\nu (\partial_k \overset{j}{e}_\lambda) - T_{i\ell} \overset{i}{e}_\nu \delta^\ell_j (\partial_\mu \overset{j}{e}_\beta)]$$

But, the second and third class of right hand side of (2.6) are vanish. That is

$$(2.7)\quad \frac{\partial T_{\nu\lambda}}{\partial x^\mu} - T_{\omega\lambda}\left\{{\omega \atop \nu\mu}\right\} - T_{\nu\omega}\left\{{\omega \atop \lambda\mu}\right\} = \frac{\partial T_{ij}}{\partial y^k} - T_{\ell j}\left\{{\ell \atop ik}\right\} - T_{i\ell}\left\{{\ell \atop jk}\right\} \overset{i}{e}_\nu \overset{j}{e}_\lambda \overset{k}{e}_\mu .$$

Hance (2.7) is equivalent to (2.2).

Corollary 2.2. We have

$$(2.8)\quad T_{ij,k} = \left[\frac{\partial T_{\nu\lambda}}{\partial x^\mu} - T_{\omega\lambda}\left\{{\omega \atop \nu\mu}\right\} - T_{\nu\omega}\left\{{\omega \atop \lambda\mu}\right\}\right] \underset{i}{e}^\nu \underset{j}{e}^\lambda \underset{k}{e}^\mu .$$

Proof. By means of (1.1), (1.2) and (2.7), we have (2.8), where

$$(2.9)\quad T_{ij,k} = \frac{\partial T_{ij}}{\partial y^k} - T_{\ell j}\left\{{\ell \atop ik}\right\} - T_{i\ell}\left\{{\ell \atop ik}\right\}$$

III. Covariant Differentiation of the Nonholonomic Contravariant and Mixtensors in V_n

The purpose of the present section is to investigate some relation between two tensor field $T^{\nu\lambda}$ and T^{ij}.

Theorem 3.1. The covariant derivative of the holonomic contravariant tensor $T^{\nu\lambda}$ may be expressed in terms of the components of nonholonomic contravariant tensors

$$(3.1)\quad T^{\nu\lambda}{}_{|\mu} = \left[\frac{\partial T^{ij}}{\partial y^k} + T^{i\ell}\left\{ {i \atop \ell k} \right\} + T^{jm}\left\{ {i \atop mk} \right\}\right] \underset{i}{e}^{\nu}\, \underset{j}{e}^{\lambda}\, e^{k}_{\mu}.$$

Proof. Similary methods of the above solution of (2.2), from (1.2) and (2.1), we have

$$(3.2)\quad \frac{\partial T^{\nu\lambda}}{\partial x^\mu} = \frac{\partial T^{ij}}{\partial y^k}\, e_i^{\nu}\, \underset{j}{e}^{\lambda}\, e^{k}_{\mu} + T^{ij}\left(\frac{\partial}{\partial y^k}\underset{i}{e}^{\nu}\right)\underset{j}{e}^{\lambda}\, e^{k}_{\mu} + T^{ij}\left(\frac{\partial}{\partial y^k}\underset{j}{e}^{\lambda}\right)\underset{i}{e}^{\nu}\, e^{k}_{\mu}.$$

By means of (1.2) and (1.4)b, we obtain

$$(3.3)\quad T^{\nu\omega}\left\{ {\lambda \atop \omega\mu} \right\} = T^{i\ell}\left\{ {i \atop \ell k} \right\} e_i^{\nu}\, \underset{j}{e}^{\lambda}\, e^{k}_{\mu} + T^{i\ell}(\partial_\mu \overset{\ell}{e}_{\omega})\, \underset{i}{e}^{\nu}\, \underset{\ell}{e}^{\omega}\, \underset{\ell}{e}^{\lambda}$$

$$(3.4)\quad T^{\lambda\theta}\left\{ {\lambda \atop \theta\mu} \right\} = T^{jm}\left\{ {i \atop mk} \right\} \underset{i}{e}^{\nu}\, \underset{j}{e}^{\lambda}\, e^{k}_{\mu} + T^{jm}(\partial_\mu \overset{m}{e}_{\theta})\, \overset{m}{e}_{\theta}\, \underset{j}{e}^{\lambda}\, \underset{m}{e}^{\nu}$$

However, the second terms of right hand side of (2.11) and (2.12), by using (1.3) and properties

$(3.5)\quad T^{\nu\lambda} = T^{\nu\mu}\delta^{\lambda}_{\mu}$, are given by

$$(3.6)\quad T^{i\ell}(\partial_\mu \overset{\ell}{e}_{\omega})\, \underset{j}{e}^{\nu}\, \underset{\ell}{e}^{\omega}\, \underset{\ell}{e}^{\lambda} = -T^{ij}(\partial_k \underset{j}{e}^{\lambda})\, \underset{i}{e}^{\nu}\, e^{k}_{\mu}$$

$$(3.7)\quad T^{jm}(\partial_\mu \overset{m}{e}_{\theta})\, \underset{m}{e}^{\theta}\, \underset{j}{e}^{\lambda}\, \underset{m}{e}^{\nu} = -T^{ij}(\partial_k \underset{i}{e}^{\nu})\, \underset{j}{e}^{\lambda}\, e^{k}_{\mu}$$

Hence, the sum of these three equations (2.10), (2.11) and (2.12) is given by

$$(3.8)\quad \frac{\partial T^{\nu\lambda}}{\partial x^\mu} + T^{\nu\omega}\left\{ {\lambda \atop \omega\mu} \right\} + T^{\lambda\theta}\left\{ {\nu \atop \theta\mu} \right\} = \frac{\partial T^{ij}}{\partial y^k} + T^{il}\left\{ {j \atop lk} \right\} + T^{jm}\left\{ {i \atop mk} \right\} \underset{i}{e}^{\nu}\, \underset{j}{e}^{\lambda}\, e^{k}_{\mu}$$

Making use of (2.9), we obtain the derivative of the nonholonomic contravariant tensor.

Corollary 3.2. We have

$$(3.9)\quad T^{ij}{}_{,k} = \left[\frac{\partial T^{\nu\lambda}}{\partial x^\mu} + T^{\nu\omega}\left\{ {\lambda \atop \omega\mu} \right\} + T^{\lambda\theta}\left\{ {\cdot\ \nu \atop \theta\mu} \right\}\right] \underset{i}{e}^{\nu}\, \underset{j}{e}^{\lambda}\, e^{k}_{\mu}.$$

Proof. In order the prove (3.9), multiplying $\overset{i}{e}_{\nu}\overset{j}{e}_{\lambda}\overset{k}{e}_{\mu}$ to both side of (3.1) and making use of (1.1) and (1.2), we have the result, where

$$(3.10)\quad T^{ij}{}_{,k} = \frac{\partial T^{ij}}{\partial y^k} + T^{ij}\left\{ {j \atop lk} \right\} + T^{im}\left\{ {i \atop mk} \right\}$$

Theorem 3.3. The covariant derivative of the holonomic mixed tensor T^{ν}_{λ} may be expressed in terms of the nonholonomic components.

$$(3.11)\quad T^{\nu}_{\lambda,\mu} = \left[T^{j}_{j}\left\{ {i \atop lk} \right\} + T_{j}\left\{ {i \atop mk} \right\} \lambda - T^{i}_{l}\left\{ {l \atop jk} \right\}\right] \underset{i}{e}^{\nu}\, e^{j}_{k}\, e^{k}_{\mu}.$$

Proof. Using the properties of (2.2) and (3.1) and making use of (1.2) and (2.1)

$$(3.12)\ \frac{\partial T^{\nu}_{\lambda}}{\partial x^{\mu}} = \left(\frac{\partial T^{i}_{j}}{\partial y^{k}}\right) e^{\nu}_{i}\, e^{j}_{\lambda}\, e^{k}_{\mu} + T^{i}_{j}\left(\frac{\partial}{\partial y^{k}} e^{\nu}_{i}\right) e^{k}_{\mu}\, e^{j}_{\lambda} + T^{i}_{j}\left(\frac{\partial}{\partial y^{k}} e^{j}_{\lambda}\right) e^{\nu}_{i}\, e^{j}_{\lambda}\, e^{k}_{\mu}$$

By virtiue of (3.5),

$$(3.13)\quad T^{\omega}_{\lambda}\left\{{\nu \atop \omega\mu}\right\} = T^{\ell}_{j}\left\{{i \atop \ell k}\right\} e^{\nu}_{i}\, e^{j}_{\lambda}\, e^{k}_{\mu} - T^{j}_{j}(\partial_{k} e^{\nu}_{i})\, e^{j}_{\lambda}\, e^{k}_{\mu}$$

$$(3.14)\quad T^{\nu}_{\omega}\left\{{\omega \atop \lambda\mu}\right\} = T^{i}_{j}\left\{{\ell \atop jk}\right\} e^{\nu}_{i}\, e^{i}_{\lambda}\, e^{k}_{\mu} - T^{i}_{j}(\partial_{k}\, e^{j}_{\lambda})\, e^{\nu}_{i}\, e^{k}_{\mu}.$$

If from the sum of these two equations (3.12) and (3.13) subtract (3.14), we have (3.15)

$$(3.15)\quad \frac{\partial T^{\nu}_{\lambda}}{\partial x^{\mu}} + T^{\omega}_{\lambda}\left\{{\nu \atop \omega\mu}\right\} - T^{\nu}_{\omega}\left\{{\omega \atop \lambda\mu}\right\}$$

$$= \left[\frac{\partial T^{i}_{j}}{\partial y^{k}} + T^{j}_{j}\left\{{i \atop lk}\right\} - T^{i}_{\ell}\left\{{\ell \atop jk}\right\}\right] e^{\nu}_{i}\, e^{j}_{\lambda}\, e^{k}_{\mu}$$

Corollary 3.4. The covariant derivative oı the nonholonomic mixed tensor T is given by

$$(3.16)\quad T^{i}_{j,k} = \left[\frac{\partial T^{\nu}_{\lambda}}{\partial x^{\mu}} + T^{\omega}_{\lambda}\left\{{\nu \atop \omega\mu}\right\} - T^{\nu}_{\omega}\left\{{\omega \atop \lambda\omega}\right\}\right] e^{i}_{\nu}\, e^{\lambda}_{j}\, e^{\mu}_{k}.$$

Proof. (3.16) follow easily from (3.11) by using of (1.1) and (1.2).

The convariant derivative of the holonomic fundamental tensors $H_{\lambda\mu}$, $H^{\lambda\mu}$ and δ^{μ}_{λ} are equivalent to zero.

Making use of the (2.3), (3.4) and (3.16), we have

Corollary 3.5. The covariant derivative of the nonholonomic fundamental tensor H_{ij}, H^{ij} and δ^{i}_{j} all vanish identically.

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國 文 抄 錄

Riemann공간 V_n 에서 Holonomic vector들의 derivative에 관한 여러가지 성질들은 이미 잘 알려진 사실이다.

본 論文에서는 Nonholonomic구조를 정의하고, 이러한 구조하에서 Nonholonomic derivative에 관한 몇가지 성질들을 Holonomic구조를 이용하여 재 구성하고 연구한다.