Continental Shelf Waves over a Double Shelf*

*Pang Ig-chan***

양향성 대륙붕의 대륙붕파*

방 익 찬**

Summary

Over a double shelf, continental shelf waves also propagate with the shallow water to the right in the Northern Hemisphere. A double shelf topography allows the existence of two sets of waves propagating in opposite directions. Their group velocities have the same direction as phase velocity in the long wave case but the opposite direction in the short wave case. The each shelf mode has a zero group velocity at some intermediate value of wave length.

Introduction

Since the advent of the continental shelf wave theory by Buchwald and Adams (1968), the theory of coastally trapped waves has gradually been established for coastal areas that lie next to a deep ocean. The general properties of continental waves over various monotonic depth profiles have been reviewed by LeBlond and Mysak (1978) and a general theory of these waves has been discussed by Huthnance (1975, 1978).

There are other coastal areas in which the depth of the ocean does not increase monotonically away from the shore. The bottom slope is reversed, for example, across submarine banks and trenches. This also admits trapped waves and these waves has been investigate by Louis (1978), Mysak *et al* (1979, 1980, 1981), and Brink (1983). The bottom slope is also reversed across a double shelf topography such as in the Yellow Sea. Although the coastally trapped waves over a double shelf topography share some of the characteristics of waves found over banks and trenches, the former differs from the latter in important dynamics ways. In spite of a basic establishment of the wave theory over a double shelf topography (Pang, 1978 : Hsueh and Pang, 1989), it still needs to develop the theory for clarifying the intrinsic dynamics. The purpose of this paper is to establish the theory of the free waves over a exponential

* 이 논문은 1989~1991 과학재단 신진연구비에 의해 연구되었음.
** 해양과학대학 해양학과

double shelf topography clearly.

Exponential bottom topography allows a simple analytical solution to the coastally trapped wave problem which can readily be compared to results from existing theories. Thus, the first step is to establish the theory with exponential bottom topography. We want to see if the theory recovers the familiar results for the single shelf case when the problem is reduced to that of two dynamically separate shelves.

Field Equation and Boundary conditions

Small perturbations to a barotropic ocean satisfy the equation

$$Hp_{xxt}+H_xp_{xt}+Hp_{yyt}+fH_xp_y+(rp_x)_x-((f^2-\omega^2)/g)p_t=-((f^2-\omega^2)/g)p_{at}+f(Y_x-X_y). \quad (1)$$

In this equation, x, y, t, p, g, f, r, H, p_a, X and Y refer, respectively, to cross-shelf distance, alongshore distance, time, the perturbation pressure divided by mean water density, the acceleration due to gravity, the Coriolis parameter, the bottom resistance coefficient, the water depth, the atmospheric pressure divided by the mean water density, the kinematic stresses in x and y direction at surface (the wind stress divided by the mean water density). The subscripts indicate the derivatives

To begin with, an intervening region is put between the two shelves, that is, the shelf 1, intervening region and shelf 2 are placed, respectively, in $-B_1<x<0$, in $0<x<L_m$, in $L_m<x<B_2$. At the coasts, no-flux boundary conditions are applied at $x=-B_1$, B_2, where the depth is three times the Ekman layer e-folding scale (Mitchum and Clarke, 1986). That is, the depth integrated offshore velocity component vanishes at a distance from the coast. Also, 'continuous pressure' and 'continuous transverse velocity' boundary conditions are applied at $x=0$, L_m. The applied boundary conditions are summarized as follows :

$$P_{1xt}+(r/h)p_{1x}+fP_{1y}=fY/h, \quad \text{at } x=-B_1 \quad (2\text{-}1)$$
$$P_1=P_m, \quad \text{at } x=0 \quad (2\text{-}2)$$
$$P_{1xt}+fP_{1y}=P_{mxt}+fP_{my}, \quad \text{at } x=0 \quad (2\text{-}3)$$
$$P_m=P_2, \quad \text{at } x=L_m \quad (2\text{-}4)$$
$$P_{mxt}+fP_{my}=P_{2xt}+fP_{2y}, \quad \text{at } x=L_m \quad (2\text{-}5)$$
$$P_{2xt}+(r/h)p_{2x}+fP_{2y}=fY/h, \quad \text{at } x=B_2 \quad (2\text{-}6)$$

Dispersion Relation

Fig.1 shows a schematic representation of the coordinate system and the geometry of two shelves (1 and 2) of exponential depth profile and a level intervening region. For convinence, let's take the positive x direction eastward. Then, the positive y direction is northward. The bottom topography (H) can be set as

$$H(x)\begin{cases} H_1 = H_0\exp(2bx), & -B_1\le x\le 0 \quad \text{in shelf 1} \\ H_m = H_m = H_0, & 0\le x\le L_m \quad \text{in middle area} \\ H_2 = H_0\exp(-2d(x-L_m)), & L_m\le x\le B_2 \quad \text{in shelf 2} \end{cases} \quad (3)$$

where b and d are the bottom slope coefficients of shelves 1 and 2.

In order to compare this result with those of existing theories, the horizontal divergence and bottom friction are not included here. The equation (1) is reduced for non-divergent, low

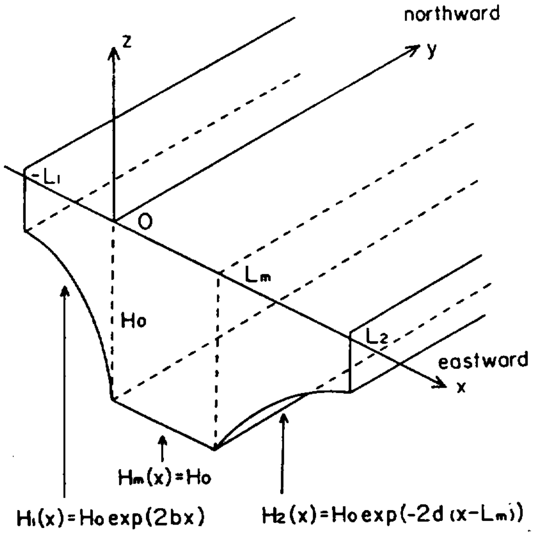

Fig. 1. Schematic representation of the coordinate system and geometry of two shelves of exponential depth profile and a level intervening region. The coordinates x, y, and z refer to the cross-shelf, alongshore, and vertical directions and are oriented eastward, northward, and upward, respectively.

-frequency long free waves in a frictionless barotropic flow as follows :

$$Hp_{xxt}+H_xp_{xt}+Hp_{yyt}+fH_xp_y=0 \qquad (4)$$

Upon substituting for the pressure, $p=F(x)\exp(i\{ly+\omega t\})$, (4) yields

$$HF''+H'F'-\ell^2HF+(f/c)H'F=0 \qquad (5)$$

where the 'prime' means the derivative with respect to x and $c=\omega/\ell$. (5) with the depth profiles given by (3) yields the following eigen value problems for the frictionless eigenfunction $F(x)$:

$$F_1''+2bF_1'+(2bf/c-\ell^2)F_1=0, \quad -B_1\le x\le 0 \qquad (6\text{-}1)$$

$$F_m''-l^2F_m=0, \quad 0\le x\le L_m \qquad (6\text{-}2)$$

$$F_2''-2dF_2'-(2df/c+\ell^2)F_2=0, \quad L_m\le x\le B_2 \qquad (6\text{-}3)$$

$$F_1'+(f/c)F_1=0, \quad \text{at } x=-B_1 \qquad (7\text{-}1)$$

(7-4)

$$F_1=F_m, \quad \text{at } x=0 \qquad (7\text{-}2)$$

$$F_1'=F_m', \quad \text{at } x=0 \qquad (7\text{-}3)$$

$$F_m=F_2, \quad \text{at } x=L_m \qquad (7\text{-}4)$$

$$F_m'=F_2', \quad \text{at } x=L_m \qquad (7\text{-}5)$$

$$F_2'+(f/c)F_2=0, \quad \text{at } x=B_2 \qquad (7\text{-}6)$$

where F_1, F_2 and F_2 represent the eigenfunctions over, respectively, the shelf 1, intervening region and shelf 2. From (6) and (7), we get the following dispersion relation with b, d, B_1, B_2, and L_m as parameters :

$$\begin{aligned}
&(-n_1-b+\ell)(n_2-d+\ell)\exp(-\ell L_m)\exp(n_1B_1)\exp(-n_2\{B_2-L_m\})\\
&+(-n_1-b+\ell)(n_2+d-\ell)\exp(-\ell L_m)\exp(n_1B_1)\exp(+n_2\{B_2-L_m\})\\
&-(n_1-b+\ell)(n_2-d+\ell)\exp(-\ell L_m)\exp(-n_1B_1)\exp(-n_2\{B_2-L_m\})\\
&-(n_1-b+\ell)(n_2+d-\ell)\exp(-\ell L_m)\exp(-n_1B_1)\exp(+n_2\{B_2-L_m\})\\
&-(n_1+b+\ell)(-n_2+d+\ell)\exp(\ell L_m)\exp(n_1B_1)\exp(-n_2\{B_2-L_m\})\\
&-(n_1+b+\ell)(-n_2-d-\ell)\exp(\ell L_m)\exp(n_1B_1)\exp(+n_2\{B_2-L_m\})\\
&+(-n_1+b+\ell)(-n_2+d+\ell)\exp(\ell L_m)\exp(-n_1B_1)\exp(-n_2\{B_2-L_m\})\\
&+(-n_1+b+\ell)(-n_2-d-\ell)\exp(\ell L_m)\exp(-n_1B_1)\exp(+n_2\{B_2-L_m\})\\
&=0
\end{aligned} \qquad (8)$$

where $n_1=[b^2-(2bf/c-\ell^2)]^{1/2}$ (9-1)

$n_2=[d^2+(2bf/c+\ell^2)]^{1/2}$ (9-2)

Phase Speed

In order to have non-trivial solution, n_1 or n_2 must be imaginary. When n_1 is imaginary, $b^2-2bf/c+\ell^2<0$. The frequency ω must obey, for a positive wave number, the inequalty, $0<\omega/f<2b\ell/(b^2+\ell^2)$. The phase speed c is thus positive in the Northern Hemisphere, which implies a southward propagation of waves. In the above inequality, ω/f goes to zero as ℓ goes both to zero and to infinity. Thus each shelf wave mode has a zero group velocity at some intermediate value of ℓ. Similarly, imaginary n_2 provides the inequality, $-2d\ell/(d^2+\ell^2)<\omega/f<0$. This gives a negative c which implies northward phase propagation. For fixed values of the parameters, we can thus find the real solution $\omega_{mn}(\ell)$, m (shelf) = 1 and 2, n (mode) = 1, 2, ⋯, of the dispersion relation (10). The solutions can be ordered, for a fixed wave number, as

$$-2d\ell/(d^2+\ell^2)<\omega_{21}/f<\omega_{22}/f<\omega_{23}/f<\cdots<0<\cdots$$
$$\cdots\omega_{13}/f<\omega_{12}/f>\omega_{11}/f<2b\ell/(b^2+\ell^2).$$

The lower the mode, the larger the absolute phase speed. Thus, one set of waves propagates northward and the other propagates southward. These are comparable to the trench waves (Mysak et al., 1979, 1981) and bank waves (Brink, 1983).

When L_m goes to infinity, the equation (8) yieds the dispersion relations for two independent shelf waves. Each of the two dispersion relations is exactly the same as one obtained by Buchwald and Adams (1968) for a single shelf adjacent to a deep ocean region of constant depth.

In the case without a central region, the dispersion relation (8) shows the dependence of the waves on the bottom topography of both shelves. It shows the constraint of the topography of one shelf on the propagtion characteristics of the shelf waves over the other.

Fig.2 and show the phase speeds of the first 10 modes in the two sets of shelf waves. One set is propagating southward along the shelf 1 and the other set is propagating northward along the shelf 2. The shelf widths used in this calculation are 400㎞ for shelf 1

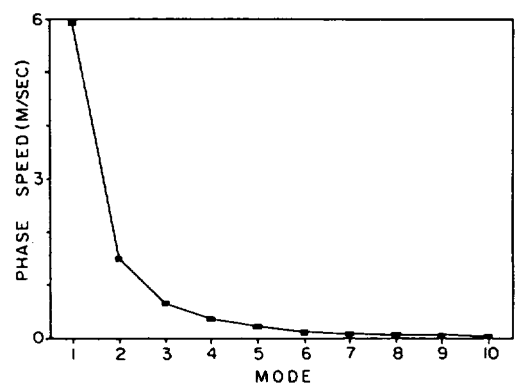

Fig. 2. Phase speeds of the first 10 modes of shelf waves propagating southward.

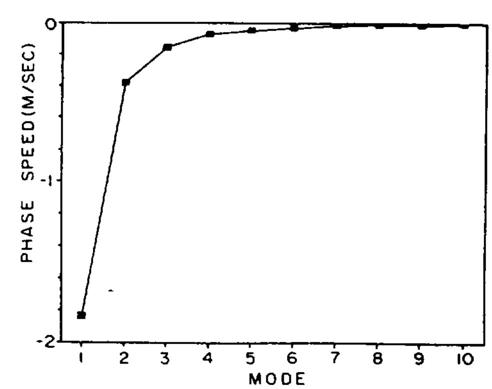

Fig. 3. Phase speeds of the first 10 modes of shelf waves propagating northward.

and 80㎞ for shelf 2 as in Fig.4. The first mode has the maximum phase speed and thereafter the phase speed decreases in higher mode. Since the phase speed of shelf waves is in some way propotional to shelf width, the phase speed of the southward propagating waves is larger than that of the northward propagating waves.

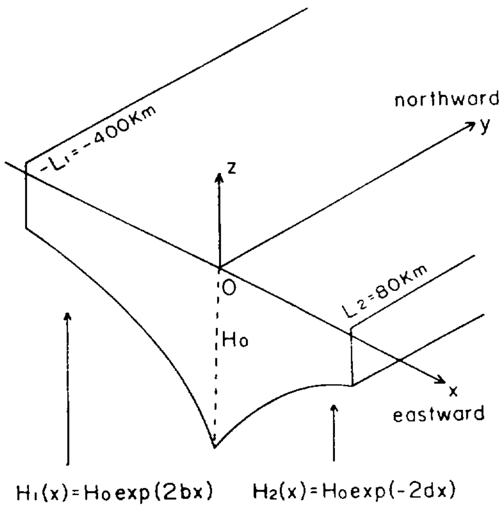

Fig. 4. Schematic representation of the coordinate system and geometry of a exponential double shelf topography. The shelf widths are 400㎞ for shelf 1 and 40㎞ for shelf 2.

Eigenfunctions

The eigenfuntions for a eigenvalue problem (6)-(7) are as follows :

$$F=\begin{cases} F_1 & \text{in shelf 1} \\ F_m & \text{in a middle area} \\ F_2 & \text{in shelf 2} \end{cases}$$

Here,

$$F_1=A\frac{a1\cdot\exp(\lambda L_2)\cdot(n_2+b1)\div a2\cdot\exp(-\lambda L_2)\cdot(n_2+b2)}{a2\ \exp(-\lambda L_2)\cdot(n_2+b2)}$$

$$\times\exp(-bx)\frac{n_1\cdot\cosh n_1(x+L_1)+(b-\frac{1}{c})\cdot\sinh n_1(x+L_1)}{n_1\cdot\cosh n_1L_1+(b-\frac{1}{c})\cdot\sinh n_1L_1}.$$

$$F_m=A\frac{a1\cdot\exp(-\lambda x)\cdot(n_2+b1)+a2\cdot\exp\lambda x\cdot(n_2+b2)}{a2\cdot(n_1+a3\cdot\tanh n_1L_1)}$$

$$F_2=A\frac{a2\cdot\exp(\lambda L_2)\cdot(n_1+c1)+a1\cdot\exp(-\lambda L_2)\cdot(n_1+c2)}{a1\cdot\exp(\lambda L_2)\cdot(n_1+c1)}$$

$$\times\exp(-dx)\frac{n_2\cdot\cosh n_2(L_m-x)+(d+\frac{1}{c})\cdot\sinh n_2(L_2-x)}{n_2\cdot\cosh n_2(L_m-L_2)+(d+\frac{1}{c})\cdot\sinh n_2(L_m-L_2)}$$

where $a1=\lambda+1/c$, $a2=\lambda-1/c$, $a3=b+\lambda$, $b1=(d+\lambda)\tanh n_2(L_m-L_2)$, $b2=(d-\lambda)\tanh n_2(L_m-L_2)$, $c1=(b+\lambda)\tanh n_1L_1$, $c2=(b-\lambda)\tanh n_1L_1$. A and λ are, respectively, a arbitrary constant and the Rossby Deformation Radius.

Fig.5 shows the amplitudes of the first 2 eigenfunctions across the shelves in two sets of shelf waves. The first eigenfunction has 1 node across the shelf and the next mode has 2 nodes and so on. It should be noted that, in the case of single shelf case, the first mode does not have any node (Clarke and

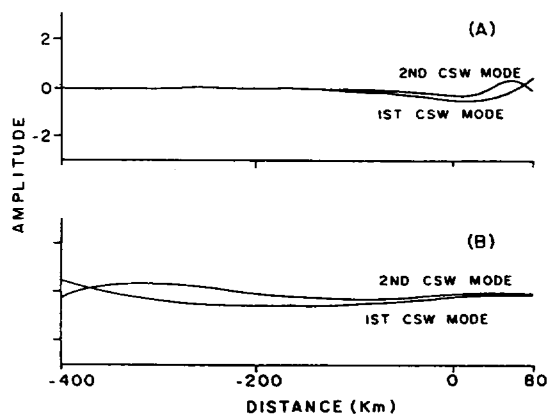

Fig. 5. **The amplitudes of the first 2 eigenfunctions of (A) shelf waves propagating northward and (B) shelf waves propagating southward. The shelf widths are 400㎞ for shelf 1 and 40㎞ for shelf 2.**

VanGoder, 1986). The southward (northward) propagating waves oscillate over the shelf 1 (shelf 2) and extend in an exponentially decay over the shelf 2 (shelf 1).

Conclusion

(1) A double shelf topography allows the existence of two sets of waves propagating in opposite directions. In the case that two shelves are apart sufficiently enough, the solutions show two independent sets of waves. However, in a double shelf case in which two shelves are adjoining each other, the waves become dependent on the geometry of both shelves.

(2) Even over a double shelf topography, shelf waves propagate with the shallow water to the right in the Northern Hemisphere. However, the group velocity of shelf wave has the same direction as phase velocity in the long wave case, but the opposite direction in the short wave case. Thus, the each shelf mode has a zero group velocity at some intermediate value of wave length.

(3) The first eigenfunction over a double shelf topography has 1 node, while the first mode over a single shelf topography does not have any node. The amplitude of shelf waves oscillate over one shelf and extend in an exponentially decay over the other shelf.

References

Brink, K.H., 1983. Low-frequency free wave and wind-driven motions over a submarine bank. *J. Phys. Oceanogr.*, 13, 103~116.

———, and J.S. Allen, 1978. On the effect of bottom friction on barotropic motion over the continental shelf. *J. Phys. Oceanogr.*, 8, 919~922.

Buchwald, V.T., and J.K. Adams, 1968. The propagation of continental shelf waves. *Proc. Roy. Soc. London*, A305, 235~250.

Clarke, A.J., and S. van Gorder, 1986. A method for estimating wind-driven frictional time-dependent, stratified shelf and slope water flow. *J. Phys. Oceanogr.*, 16, 1013~1028.

Gill, A.E., and E.H. Schumann, 1974. The generation of long shelf waves by the wind. *J. Phys. Oceanogr.*, 4, 83~90.

Hsueh, Y., and I.C. Pang, 1989. Coastally trapped long waves in the Yellow Sea. *J. Phys. Oceanogr.*, 19, 5, 612~625.

Huthnance, J.M., 1975. On trapped waves over a continental shelf. *J. Fluid Mech.*, 67, 689~704.

————, 1978. On coastal trapped waves; Analysis calculation by inverse iteration. *J. Phys. Oceanogr.*, 8, 74~92.

LeBlond, P.H., and L.A. Mysak, 1978. Waves in the Oceans. Elsevier, 602pp.

Louis, J.P., 1978. Low-frequency edge waves over a trenchridge topography adjoining a straight coastline. *Geophys. Astrophys. Fluid Mech.*, 55, 113~127.

Mitchum, G.T., and A.J. Clarke, 1986. The frictional nearshore response to forcing by synoptic scale winds. *J. Phys. Oceanogr.*, 16, 934~946.

Mysak, L.A., and P.H. LeBlond, andW.J. Emery, 1979. Trench waves. *J. Phys.*

Oceanogr., 9, 1001~1013.

———, 1980. Recent advences in shelf wave dynamics. *Rev. Geophys. Space Phys.*, 18, 211~241.

———, and A.J. Willmott, 1981. Forced trench waves. *J. Phys. Oceanogr.*, 11, 1481~1502.

Pang, I.C., 1987. Theory of coastally trapped waves and its application to the Yellow Sea. Ph.D. Dissertation, Florida State University, 128pp.

國文抄錄

양향성 대륙붕에서도 대륙붕파는 북반구에서 해안선을 오른쪽에 두고 진행한다. 양향성 대륙붕에서는 서로 반대 방향으로 진행하는 두 집합의 대륙붕파가 생성된다. 군속도(group velocity)는 장파인 경우에는 파의 진행방향과 같으나 단파인 경우에는 반대가 된다. 그러므로 각 대륙붕파는 어떤 중간의 파장에서 0인 군속도를 갖게 된다.