Spectral Theory for Nonlinear Maps

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Introduction

Furi and Vignoli(1975) introduced a class of all quasi-bounded (nonlinear) maps on a Banach space X and defined a spectrum for this class. They gave some of the basic properties for such spectrum, and extended surjectivity results previously obtained by Granas and Kranosel'skij.

Canavati(1979) defined a numerical range for a broader class of all numerically bounded maps on a Banach space X and studied it in a more systematic way. Kim and Yang(1984) defined a new class of all numerically bounded maps on a Hilbert C^*-module and studied their properties. What we shall do here is to define a joint numerical range for a broader class of n-tuples of continuous maps; the "jointly $*$-numerically bounded" n-tuples (to be defined below) on a Banach space, and give some properties of it. Among other properties, our joint numerical range will be compact and connected, and will coincide with the closure of joint spatial numerial range $\overline{V(\mathbf{T})}$, in the particular case when $\mathbf{T} = (T_1, \cdots, T_n)$ is an n-tuple of bounded linear operators on a Banach space. Also we define a joint asymptotic spectrum for a class of jointly $*$-quasibounded n-tuples (to be defined below) of maps, and show that it is a compact subset of our numerical range. Finally we will introduce the concept of joint lower $*$-numerical range and investigate its properties, and we are going to define the numerical range for the numerically bounded n-tuple of vector fields on the unit sphere of a Banach space. In particular, if $n = 1$, our concepts coincide with those of Canavati.

The plan of the work is as follows: In section 2 we will define some classes of n-tuples of continuous maps on a Banach space X, which are going to be the object of study in this work. For reasons that are going to be apparent in later sections, we found more convenient to deal with maps of the form $F : X \times X^* \to X$, instead of maps $f : X \to X$, the later being a particular case of the former. Here X^* denotes the dual space of X. In section 3 we will define the joint $*$-numerical range for the n-tuple $\mathbf{F} = (F_1, \cdots, F_n)$ of maps. In section 4, we are going to define the joint $*$-asymptotic spectrum of an n-tuple $\mathbf{F}$ and study its relations with the joint $*$-numerical range. In section 5, we will introduce the concept of adjoint of jointly $*$-numerically bounded n-tuple and we will show that they have the same numerical range. Finally, in section 6 we are going to define the joint numerical range for the numerically bounded n-tuple of vector fields on the unit sphere of a Banach space.

Thoughout this paper, let X be a Banach space over $\mathbf{K}$(R or C), X^* its dual space, and denote by $< x, x^* >$ $(x \in X, x^* \in X^*)$ the duality map between X and X^*. If $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbf{K}^n$, we set $|\lambda| = \left(\sum_{i=1}^{n} |\lambda_i|^2\right)^{\frac{1}{2}}$. For an n-tuple $\mathbf{F} =$

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$(F_1, \cdots, F_n)$ of maps and $x \in X$, $\mathbf{F}(x)$ means $\mathbf{F}(x) = (F_1(x), \cdots, F_n(x))$, and $< \mathbf{F}(x), x^* >$ denotes $(< F_1(x), x^* >, \cdots, < F_n(x), x^* >)$.

Some Spaces of n-tuples of Nonlinear Maps

DEFINITION 2.1. *Let X be a Banach space over the field $\mathbf{K}$. (a) $\mathbf{B}^n(X)$ is the vector space of all n-tuples $\mathbf{f} = (f_1, \cdots, f_n)$ of continuous maps $f_i : X \to X$ such that*

$$\|\mathbf{f}(x)\| = \left(\sum_{j=1}^{n} \|f_j(x)\|^2\right)^{\frac{1}{2}} \leq M\|x\|$$

for some $M \geq 0$ and all $x \in X$. We define the joint norm $\|\mathbf{f}\|$ of $\mathbf{f} = (f_1, \cdots, f_n)$ as the smallest $M \geq 0$ such that this inequality holds for all $x \in X$. An element of $\mathbf{B}^n(X)$ is called a jointly bounded n-tuple on X. (b) $\mathbf{Q}^n(X)$ is the vector space of all jointly quasibounded n-tuples on X. That is, the space of all n-tuples $\mathbf{f} = (f_1, \cdots, f_n)$ of continuous maps $f_i : X \to X$ such that there exist $A, B \geq 0$ satisfying

$$\|\mathbf{f}(x)\| = \left(\sum_{j=1}^{n} \|f_j(x)\|^2\right)^{\frac{1}{2}} \leq A + B\|x\|, \quad x \in X. \tag{1}$$

Denote $|\mathbf{f}|$ the joint quasinorm of $\mathbf{f} = (f_1, \cdots, f_n)$ to be the infimum of all $B \geq 0$ for which (1) holds for some $A \geq 0$, i.e.,

$$|\mathbf{f}| = \limsup_{\|x\| \to \infty} \frac{\|\mathbf{f}(x)\|}{\|x\|}$$

In particular, if $n = 1$, then $\mathbf{B}^n(X)$ is the vector space of all bounded maps on X and $\mathbf{Q}^n(X)$ is the vector space of all quasibounded maps on X. Notice that $\|\cdot\|$ is a norm on $\mathbf{B}^n(X)$ and $|\cdot|$ is a semi-norm on $\mathbf{Q}^n(X)$.

The norm $\times$ weak* topology in $X \times X^*$, is the topology in $X \times X^*$ given by the norm topology on X and the weak* topology on X^*(Bonsall Duncan,1971).

We define the following subsets of $X \times X^*$,

$$\Pi_r = \{ (x, x^*) \in X \times X^* : \|x\| = \|x^*\| \geq r, \ |x\|^2 = < x, x^* > \}$$

for $r > 0$, and

$$\Pi_0 = \bigcup_{r>0} \Pi_r.$$

LEMMA 2.2(BONSALL & DUNCAN, 1973). *Let π denote the natural projection of $X \times X^*$ onto X, and let E be a subset of Π_r that is relatively closed in Π_r with respect to the norm $\times$ weak* topology. Then $\pi(E)$ is a (norm) closed subset of X.*

LEMMA 2.3(BONSALL & DUNCAN, 1973). *Each $\Pi_r (r > 0)$ and Π_0 are connected subsets of $X \times X^*$ with the norm $\times$ weak* topology, unless X has dimension one over $\mathbf{R}$.*

From now on we shall assume that Π_0 has the norm $\times$ weak* topology induced as a subset of $X \times X^*$. Also we shall assume that X does not have dimension one over $\mathbf{R}$.

DEFINITION 2.4. *Let $\mathbf{F} = (F_1, \cdots, F_n)$ be an n-tuple of continuous maps from Π_0 into X. We say that $\mathbf{F}$ is jointly $*$-bounded if .*

$$\|\mathbf{F}\|_* = \sup_{\Pi_*} \frac{\|\mathbf{F}(x, x^*)\|}{\|x\|} < \infty$$

We denote by $\mathbf{B}^n_*(X)$ the vector space of all jointly $*$-bounded n-tuples.

Notice that $\|\cdot\|_*$ is a norm on $\mathbf{B}^n_*(X)$. We can consider the vector space $\mathbf{B}^n(X)$ of all n-tuples of bounded maps as a vector subspace of $\mathbf{B}^n_*(X)$ in a natural way, namely ; if $\mathbf{f} = (f_1, \cdots, f_n) \in \mathbf{B}^n(X)$, the mapping $\mathbf{F}(x, x^*) = \mathbf{f}(x) = (f_1(x), \cdots, f_n(x))$ belongs to $\mathbf{B}^n_*(X)$ and $\|\mathbf{f}\| = \|\mathbf{F}\|_*$.

THEOREM 2.5. *$\mathbf{B}^n_*(X)$ is a Banach space.*

Proof. This is a standard argument, and so it will be omitted.

DEFINITION 2.6. *Let $\mathbf{F} = (F_1, \cdots, F_n)$ be an n-tuple of continuous maps from Π_0 into X. We say that $\mathbf{F}$ is jointly $*$-quasibounded if*

$$|\mathbf{F}|_* = \lim_{r \to \infty} \sup_{\Pi_r} \frac{\|\mathbf{F}(x, x^*)\|}{\|x\|} < +\infty.$$

We denote by $\mathbf{Q}^n_*(X)$, the vector space of all jointly $*$-quasibounded n-tuples.

Notice that $|\cdot|_*$ is a seminorm on $\mathbf{Q}^n_*(X)$. Obiously one has $\mathbf{B}^n_*(X) \subseteq \mathbf{Q}^n_*(X)$ and

$$|\mathbf{F}|_* \leq \|\mathbf{F}\|_*.$$

We can consider the vector space $\mathbf{Q}^n(X)$ as a vector space $\mathbf{Q}^n_*(X)$ in a natural way, namely ; if $\mathbf{f} \in \mathbf{Q}^n(X)$, then the mapping $\mathbf{F}(x, x^*) = \mathbf{f}(x)$ belongs to $\mathbf{Q}^n_*(X)$ and $|\mathbf{f}| = |\mathbf{F}|_*$.

LEMMA 2.7. *For any $\mathbf{F} \in \mathbf{Q}^n_*(X)$, there exists a* *sequence* $< \mathbf{F}_m >$ *in* $\mathbf{B}^n_*(X)$ *such that* $|\mathbf{F}_m - \mathbf{F}|_* = 0 \ (m = 1, 2, 3, \cdots)$ *and*

$$\|\mathbf{F}_m\|_* \longrightarrow |\mathbf{F}|_* \quad \text{as} \quad m \to \infty.$$

Proof. Let $\rho^2 = \|x\|^2 + \|x^*\|^2$, and define

$$\mathbf{F}_m(x, x^*) = \begin{cases} \mathbf{F}(x, x^*) & \text{if } \rho \geq m, \\ \frac{\rho}{m}\mathbf{F}(\frac{m}{\rho}x, \frac{m}{\rho}x^*) & \text{if } 0 < \rho < m. \end{cases}$$

We have

$$\|\mathbf{F}_m\|_* = \sup_{\Pi_*} \frac{\|\mathbf{F}_m(x, x^*)\|}{\|x\|} = \sup_{\Pi_{m/\sqrt{2}}} \frac{\|\mathbf{F}(x, x^*)\|}{\|x\|}.$$

Therefore $\mathbf{F}_m \in \mathbf{B}^n_*(X)$ for all m large enough and

$$\|\mathbf{F}_m\|_* \to |\mathbf{F}|_* \quad \text{as} \quad m \to \infty.$$

DEFINITION 2.8. *(a) Let* $\mathbf{F}, \mathbf{G} \in \mathbf{Q}^n_*(X)$. *The n-tuple* $\mathbf{F}$ *is said to be jointly* $*$*-asymptotically equivalent to* $\mathbf{G}$ *(j.*$*$*- a.e) if* $|\mathbf{F} - \mathbf{G}|_* = 0$. *It is easy to see that this is an equivalence relation. (b)* $\widetilde{\mathbf{Q}}^n_*(X)$ *is the normed space of all equivalence class of jointly* $*$*-quasibounded n-tuples, i.e.* $\widetilde{\mathbf{Q}}^n_*(X) = \mathbf{Q}^n_*(X)/N^n(|\cdot|_*)$, *where* $\mathbf{F} \in N^n(|\cdot|_*)$ *iff* $|\mathbf{F}|_* = 0$. *The norm on* $\widetilde{\mathbf{Q}}^n_*(X)$ *is the one induced by* $|\cdot|_*$ *and will be denoted in the same way.*

From Lemma 2.7, we see that the mapping $\mathbf{B}^n_*(X) \to \widetilde{\mathbf{Q}}^n_*(X)$, $\mathbf{F} \to \widetilde{\mathbf{F}}$ is onto.

Furthermore we have:

THEOREM 2.9. $\widetilde{\mathbf{Q}}^n_*(X)$ *is a Banach space.*

Proof. Let $\{\widetilde{\mathbf{F}}_m = < \widetilde{F}^{(1)}_m, \cdots, \widetilde{F}^{(n)}_m >\}$ be any sequence in $\widetilde{\mathbf{Q}}^n_*(X)$ such that $\sum |\widetilde{\mathbf{F}}_m|_*$ converges. We have to show that $\sum \widetilde{\mathbf{F}}_m = (\sum \widetilde{F}^{(1)}_m, \cdots, \sum \widetilde{F}^{(n)}_m)$ converges. i.e. $\sum \widetilde{F}^{(j)}_m$ converges for each $j = 1, \cdots, n$. By Lemma 2.7, for any positive integer m, we can choose $\mathbf{G}_m \in \mathbf{B}^n_*(X)$ such that

$$\widetilde{\mathbf{G}}_m = \widetilde{\mathbf{F}}_m \text{ and } \|\mathbf{G}_m\|_* \leq |\mathbf{F}_m|_* + 2^{-m}.$$

Since $\mathbf{B}^n_*(X)$ is a Banach space, $\sum \mathbf{G}_m$ converges to an element $\mathbf{G} \in \mathbf{B}^n_*(X)$. From the continuity of the linear projection $B^n_*(X) \to \widetilde{\mathbf{Q}}^n_*(X)$, we obtain $\sum \widetilde{\mathbf{G}}_m = \sum \widetilde{\mathbf{F}}_m = \widetilde{\mathbf{G}}$.

DEFINITION 2.10. *Let* $\mathbf{F} = (F_1, \cdots, F_n)$ *be an n-tuple of continuous maps from* Π_0 *into* X. *We say that* $\mathbf{F}$ *is jointly* $*$*-numerically bounded if*

$$\omega_*(\mathbf{F}) = \limsup_{r \to \infty\, \Pi_r} \frac{|< \mathbf{F}(x, x^*), x^* >|}{\|x\|\|x^*\|} < +\infty.$$

We denote by $\mathbf{W}^n_*(X)$, *the vector space of all jointly* $*$*-numerically bounded n-tuples.*

Notice that w_* is a seminorm on $\mathbf{W}^n_*(X)$. If $\mathbf{F} \in \mathbf{W}^n_*(X)$, then we let

$$\alpha_*(\mathbf{F}) = \liminf_{r \to \infty\, \Pi_r} \frac{|< \mathbf{F}(x, x^*), x^* >|}{\|x\|\|x^*\|}.$$

Obviously one has $\mathbf{Q}^n_*(X) \subseteq \mathbf{W}^n_*(X)$ and $w_*(\mathbf{F}) \leq |\mathbf{F}|_*$.

DEFINITION 2.11. *Let* $\mathbf{F} = (F_1, \cdots, F_n) \in W^n_*(X)$ *and for* $j = 1, \cdots, n$, *consider the maps*

$$F^\nu_j : \Pi_0 \to X \quad \text{and} \quad F^\tau_j : \Pi_0 \to X$$

given by

$$F^\nu_j(x, x^*) = \frac{< F_j(x, x^*), x^* >}{\|x\|\|x^*\|} x$$

and

$$F^\tau_j(x, x^*) = F_j(x, x^*) - F^\nu_j(x, x^*).$$

Then $\mathbf{F} = \mathbf{F}^\nu + \mathbf{F}^\tau$ *(i.e.,* $F_j = F^\nu_j + F^\tau_j$ *for* $j = 1, \cdots, n$.) *The n-tuples* $\mathbf{F}^\nu = (F^\nu_1, \cdots, F^\nu_n)$ *and* $\mathbf{F}^\tau = (F^\tau_1, \cdots, F^\tau_n)$ *are called the jointly normal and jointly tangent components of* $\mathbf{F}$ *respectively.*

The following Lemma follows immediately from the definitions.

LEMMA 2.12. *Let* $\mathbf{F} = (F_1, \cdots, F_n) \in \mathbf{W}^n_*(X)$. *Then*

(a) $< \mathbf{F}^\nu(x, x^*), x^* > = < \mathbf{F}(x, x^*), x^* >, \ (x, x^*) \in \Pi_0$.

(b) $< \mathbf{F}^\tau(x, x^*), x^* > = 0, \ (x, x^*) \in \Pi_0$.

(c) $\mathbf{F}^\nu \in \mathbf{Q}^n_*(X)$ *and* $|\mathbf{F}^\nu|_* = \omega_*(\mathbf{F})$.

The following result is also obvious.

THEOREM 2.13. *Let* $\mathbf{F} = (F_1, \cdots, F_n)$ *be an n-tuple of continuous maps from* Π_0 *into* X. *Then* $\mathbf{F} \in \mathbf{W}^n_*(X)$ *if and only if there exists n-tuples* $\mathbf{G}, \mathbf{H}$ *of continuous maps from* Π_0 *into* X *with* $\mathbf{G} \in \mathbf{Q}^n_*(X)$ *and* $\mathbf{H}$ *satisfying* $< \mathbf{H}(x, x^*), x^* > = 0 \ \ ((x, x^*) \in \Pi_0)$, *such that* $\mathbf{F} = \mathbf{G} + \mathbf{H}$. *Such an n-tuple* $\mathbf{H}$ *is said to be a jointly* $*$*-orthogonal n-tuple.*

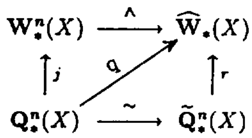
DEFINITION 2.14. *(a) Let* $\mathbf{F}, \mathbf{G} \in \mathbf{W}^n_*(X)$. *The n-tuple* $\mathbf{F}$ *is said to be jointly* $*$*-asymptotically numerically equivalent (i.e.j.* $*$*-a.n.e) to* $\mathbf{G}$ *if* $\omega_*(\mathbf{F} - \mathbf{G}) = 0$.

It is easy to see that there is an equivalence relation.

(b) $\widehat{\mathbf{W}}^n_*(X)$ *is the normed space of all equivalence classes of jointly* $*$*-numerically bounded n-*

tuples, i.e., $\widehat{\mathbf{W}}^n_*(X) = \mathbf{W}^n_*(X)/\mathbf{N}^n(\omega_*)$, *where* $\mathbf{F} \in N^n(\omega_*)$ *iff* $\omega_*(\mathbf{F}) = 0$. *The norm on* $\widehat{\mathbf{W}}^n_*(X)$ *is the one induced by* ω_*, *and it will be denoted in the same way.*

Now let $\sim$: $\mathbf{Q}^n_*(X) \to \widetilde{\mathbf{Q}}_*(X)$ and $\wedge : \mathbf{W}^n_*(X) \to \widehat{\mathbf{W}}^n_*(X)$ be natural linear projections. Then we have the following commutative diagram of continous linear maps



where j is the inclusion map of $\mathbf{Q}^n_*(X)$ into $\mathbf{W}^n_*(X)$, $q(\mathbf{F}) = \widehat{\mathbf{F}}$ and $r(\widetilde{\mathbf{F}}) = \widehat{\mathbf{F}}$.

Note that the map r is well-defined, because if $\mathbf{F}, \mathbf{G} \in \mathbf{Q}^n_*(X)$ are such that $\widetilde{\mathbf{F}} = \widetilde{\mathbf{G}}$, then $\omega_*(\mathbf{F} - \mathbf{G}) \leq |\mathbf{F} - \mathbf{G}|_* = 0$, and hence $\widehat{\mathbf{F}} = \widehat{\mathbf{G}}$.

THEOREM 2.15. $\widehat{\mathbf{W}}^n_*(X)$ *is a Banach space.*

Proof. Let $\{\widehat{\mathbf{F}}_m\}$ be a sequence in $\widehat{\mathbf{W}}^n_*(X)$ such that $\sum \omega_*(\widehat{\mathbf{F}}_m) < \infty$. We have to show that $\sum \widehat{\mathbf{F}}_m = (\sum \widehat{F}^{(1)}_m, \cdots, \widehat{F}^{(n)}_m)$ converges.

Since $\omega_*(\widehat{\mathbf{F}}) = \omega_*(\mathbf{F}) = |\mathbf{F}^\nu|_* = |\widetilde{\mathbf{F}}^\nu|_*(\mathbf{F} \in \mathbf{W}^n_*(X))$, where $\mathbf{F}^\nu \in \mathbf{Q}^n_*(X)$ (Lemma 2.12) is the jointly normal component of $\mathbf{F}$, we have

$$\sum |\widetilde{\mathbf{F}}^\nu_m|_* = \sum \omega_*(\widehat{\mathbf{F}}_m) < \infty. \qquad (1)$$

But $\{\widetilde{\mathbf{F}}^\nu_m\}$ is a sequence in the Banach space $\widetilde{\mathbf{Q}}^n_*(X)$, and it follows from (1) and Theorem 2.9 that the series $\sum \widetilde{\mathbf{F}}^\nu_m$ converges to an element $\widetilde{\mathbf{F}} \in \widetilde{\mathbf{Q}}^n_*(X)$. Since the mapping $r : \widetilde{\mathbf{Q}}^n_*(X) \to \widehat{\mathbf{W}}^n_*(X)$ is linear and continous, we must have

$$\sum \widehat{\mathbf{F}}^\nu_m = \sum r(\widetilde{\mathbf{F}}^\nu_m) = r(\widetilde{\mathbf{F}}) = \widehat{\mathbf{F}}. \qquad (2)$$

But $\widehat{\mathbf{F}} = \widehat{\mathbf{F}}^\nu$ for $\mathbf{F} \in \mathbf{W}^n_*(X)$. Hence from (2) we obtain $\sum \widehat{\mathbf{F}}_m = \widehat{\mathbf{F}}$.

The joint $*$-numerical range

DEFINITION 3.1. *Let* $\mathbf{F} \in W^n_*(X)$ *and consider the continuous map* $\phi_{\mathbf{F}} : \Pi_0 \to \mathbf{K}^n$ *given by*

$$\phi_{\mathbf{F}}(x, x^*) = \frac{< \mathbf{F}(x, x^*), x^* >}{\|x\|\|x^*\|}.$$

We define the joint $*$*-numerical range* $\Omega_*(\mathbf{F})$ *of* $\mathbf{F} = (F_1, \cdots, F_n)$ *as the set*

$$\Omega_*(\mathbf{F}) = \bigcap_{r>0} \overline{\phi_{\mathbf{F}}(\Pi_r)}.$$

In other words, $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Omega_*(\mathbf{F})$ *if and only if there exists a sequence* $\{(x_k, x^*_k)\}$ *in* Π_0 *such that* $\|x_k\| \geq k$ *and*

$$\frac{< F_j(x_k, x^*_k), x^*_k >}{\|x_k\|\|x^*_k\|} \to \lambda_j \quad \text{as} \quad k \to \infty \qquad (j = 1, \cdots, n).$$

THEOREM 3.2. *If* $\mathbf{F} \in W^n_*(X)$, *then* $\Omega_*(\mathbf{F})$ *is a nonempty compact connected subset of* $\mathbf{K}^n$.

Proof. Since $\mathbf{F} \in W^n_*(X)$, the set $\overline{\phi_{\mathbf{F}}(\Pi_r)}$ are bounded for all $r > 0$ large enough. Now $\{\overline{\phi_{\mathbf{F}}(\Pi_r)}\}$ is a nest family of compact nonempty sets, therefore by Cantor's theorem $\Omega_*(\mathbf{F}) \neq \phi$ and is compact.

Now from Lemma 2.3, we have that each $\overline{\phi_{\mathbf{F}}(\Pi_r)}$ is a connected subset of $\mathbf{K}^n$. Thus $\Omega_*(\mathbf{F})$ being intersection of a nested family of compact connected sets is connected as well (Kuratowsky, 1973).

It is obvious that for any $\mathbf{F}, \mathbf{G} \in W^n_*(X)$ and $\mu \in \mathbf{K}$,

(a) $\Omega_*(\mu\mathbf{F}) = \mu\Omega_*(\mathbf{F})$ and

(b) $\Omega_*(\mathbf{F} + \mathbf{G}) \subseteq \Omega_*(\mathbf{F}) + \Omega_*(\mathbf{G})$,

where $\mu\mathbf{F}$ denotes $(\mu F_1, \cdots, \mu F_n)$.

REMARK 3.3. *For any* $\mathbf{F}, \mathbf{G} \in W^n_*(X)$ *and* $\mu \in \mathbf{K}$,

(a) $\mathbf{F}^\nu \in Q^n_*(X)$ *and* $|\mathbf{F}^\nu|_* = w_*(\mathbf{F})$,
(b) $\Omega_*(\mathbf{F}^\nu) = \Omega_*(\mathbf{F})$ *and* $\Omega_*(\mathbf{F}^r) = \{\mathbf{o}\}$,
(c) $\Omega_*(\mu\mathbf{F}) = \mu\Omega_*(\mathbf{F})$, *and*
(d) $\Omega_*(\mathbf{F} + \mathbf{G}) \subseteq \Omega_*(\mathbf{F}) + \Omega_*(\mathbf{G})$,

THEOREM 3.4. *If* $\mathbf{F}, \mathbf{G} \in W^n_*(X)$ *and* $w_*(\mathbf{F} - \mathbf{G}) = 0$, *then* $\Omega_*(\mathbf{F}) = \Omega_*(\mathbf{G})$.

Proof. From the above remark, we have

$$\Omega_*(\mathbf{F}) = \Omega_*(\mathbf{F}^\nu) \quad \text{and} \quad \Omega_*(\mathbf{G}) = \Omega_*(\mathbf{G}^\nu).$$

Also we have $|\mathbf{F}^\nu - \mathbf{G}^\nu|_* = w_*(\mathbf{F} - \mathbf{G}) = 0$.

We shall show that $\Omega_*(\mathbf{F}^\nu) = \Omega_*(\mathbf{G}^\nu)$. Let $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Omega_*(\mathbf{F}^\nu)$. Then there is a sequence $\{(x_k, x^*_k)\}$ in Π_0 such that $\|x_k\| \geq k$ and

$$\frac{< F^\nu_j(x_k, x^*_k), x^*_k >}{\|x_k\|\|x^*_k\|} \to \lambda_j \quad \text{as} \quad k \to \infty (j = 1, \cdots, n).$$

Now $\dfrac{< G^\nu_j(x_k, x^*_k), x^*_k >}{\|x_k\|\|x^*_k\|} =$

$$\frac{< (G^\nu_j - F^\nu_j)(x_k, x^*_k), x^*_k >}{\|x_k\|\|x^*_k\|} + \frac{< F^\nu_j(x_k, x^*_k), x^*_k >}{\|x_k\|\|x^*_k\|} \qquad (1)$$

But

$$\frac{|<(G_j^\nu - F_j^\nu)(x_k, x_k^*), x_k^*>|}{\|x_k\|\|x_k^*\|} \le \frac{\|(G_j^\nu - F_j^\nu)(x_k, x_k^*)\|}{\|x_k\|},$$

and $|G^\nu - F^\nu|_* = 0$ imply

$$\frac{<(G_j^\nu - F_j^\nu)(x_k, x_k^*), x_k^*>}{\|x_k\|\|x_k^*\|} \to 0 \quad (j = 1, 2, \cdots, n). \tag{2}$$

Hence from (1) and (2), we see that

$$\frac{<G_j^\nu(x_k, x_k^*), x_k^*>}{\|x_k\|\|x_k^*\|} \to \lambda_j \quad \text{as} \quad k \to \infty \quad (j = 1, 2, \cdots, n).$$

Therefore $\Omega_*(\mathbf{F}^\nu) \subseteq \Omega_*(\mathbf{G}^\nu)$. The inclusion $\Omega_*(\mathbf{G}^\nu) \subseteq \Omega_*(\mathbf{F}^\nu)$ is proved in the same way.

THEOREM 3.5. *If* $\mathbf{F} \in W_*^n(X)$, *then* $\alpha_*(\mu\pi - \mathbf{F}) \ge dist(\mu, \Omega_*(\mathbf{F}))$, $\mu = (\mu_1, \cdots, \mu_n) \in \mathbf{K}^n$, *where* $\mu\pi = (\mu_1\pi, \cdots, \mu_n\pi)$.

Proof. We shall show a little more, namely; that for any $\mu = (\mu_1, \cdots, \mu_n) \in \mathbf{K}^n$, there exists $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Omega_*(\mathbf{F})$ such that $\alpha_*(\mu\pi - \mathbf{F}) = |\mu - \lambda|$.

By definition of $\alpha_*(\mu\pi - \mathbf{F})$, there is a sequence $\{(x_k, x_k^*)\}$ in Π_0 such that $\|x_k\| \ge k$ and

$$\frac{|<(\mu\pi - \mathbf{F})(x_k, x_k^*), x_k^*>|}{\|x_k\|\|x_k^*\|} \to \alpha_*(\mu\pi - \mathbf{F}). \tag{1}$$

Since $\mathbf{F} \in W_*^n(X)$, without loss of generality we may assume that the sequence.

$$\left\{\frac{<\mathbf{F}(x_k, x_k^*), x_k^*>}{\|x_k\|\|x_k^*\|}\right\}$$

is convergent to some $\lambda \in \Omega_*(\mathbf{F})$.

Thus from (1) we obtain

$$\begin{aligned}
&\lambda_*(\mu\pi - \mathbf{F}) \\
&= \lim_{k\to\infty} \frac{|<\mu\pi(x_k, x_k^*), x_k^*> - <\mathbf{F}(x_k, x_k^*), x_k^*>|}{\|x_k\|\|x_k^*\|} \\
&= \lim_{k\to\infty} \left| \frac{<x_k, x_k^*>\mu}{\|x_k\|\|x_k^*\|} - \frac{<\mathbf{F}(x_k, x_k^*), x_k^*>}{\|x_k\|\|x_k^*\|} \right| \\
&= |\mu - \lambda|.
\end{aligned}$$

THEOREM 3.6. *If* $\mathbf{F} \in W_*^n(X)$, *then* $\Omega_*(\mathbf{F}) = \{\lambda \in \mathbf{K}^n : \alpha_*(\lambda\pi - \mathbf{F}) = 0\}$.

Proof. Let $\Lambda = \{\lambda \in \mathbf{K}^n : \alpha_*(\lambda\pi - \mathbf{F}) = 0\}$. Then from Theorem 3.5, we have $\Lambda \subseteq \Omega_*(\mathbf{F})$. Now let $\lambda \in \Omega_*(\mathbf{F})$. Then there is a sequence $\{(x_k, x_k^*)\}$ in Π_0 such that $\|x_k\| \ge k$ and

$$\frac{<\mathbf{F}(x_k, x_k^*), x_k^*>}{\|x_k\|\|x_k^*\|} \to \lambda = (\lambda_1, \cdots, \lambda_n) \quad \text{as} \quad k \to \infty.$$

This, in turn, implies that

$$\frac{<(\lambda_j\pi - F_j)(x_k, x_k^*), x_k^*>}{\|x_k\|\|x_k^*\|} \to 0 \quad \text{as} \quad k \to \infty \quad (j = 1, 2, \cdots, n).$$

and hence that $\alpha_*(\lambda\pi - \mathbf{F}) = 0$. Therefore $\lambda \in \Lambda$ and $\Omega_*(\mathbf{F}) \subseteq A$.

REMARK 3.7. *For any* $\mathbf{F}, \mathbf{G} \in W_*^n(X)$ *and* $\mu \in \mathbf{K}$,

(a) $0 \le \alpha_*(\mathbf{F}) \le w_*(\mathbf{F})$,
(b) $\alpha_*(\mu\mathbf{F}) = |\mu|\alpha_*(\mathbf{F})$,
(c) $\alpha_*(\mathbf{F} + \mathbf{G}) \le \alpha_*(\mathbf{F}) + w_*(\mathbf{G})$,
(d) $\alpha_*(\mathbf{F}) - w_*(\mathbf{G}) \le \alpha_*(\mathbf{F} + \mathbf{G})$,
(e) $|\alpha_*(\mathbf{F}) - \alpha_*(\mathbf{G})| \le w_*(\mathbf{F} - \mathbf{G})$, *and*
(f) $\alpha_*(\mathbf{F}) \le |\lambda|$, *if* $\alpha \in \Omega_*(\mathbf{F})$.

Recall that if (M, d) is a metric space and $\Gamma(M)$ denotes the set of all non-void closed subsets of M, and if we define

$$\gamma(A, B) = \max\{\sup_{x\in B} dist(x, A), \quad \sup_{x\in A} dist(x, B)\}, \quad A, B \in \Gamma(M).$$

Then $(\Gamma(M), \gamma)$ is a metric space. The metric γ is called the Hausdorff metric.

THEOREM 3.8. *If* $\mathbf{F}, \mathbf{G} \in W_*^n(X)$, *then*

$$\gamma(\Omega_*(\mathbf{F}), \Omega_*(\mathbf{G})) \le w_*(\mathbf{F} - \mathbf{G}),$$

where γ *denotes the Hausdorff metric in* $\Gamma(\mathbf{K}^n)$.

Proof. We have

$$\begin{aligned}
\gamma(\Omega_*(\mathbf{F}), \Omega_*(\mathbf{G})) = \max\{&\sup\{dist(\lambda, \Omega_*(\mathbf{F}) : \lambda \in \Omega_*(\mathbf{G})\}, \\
&\sup\{dist(\lambda, \Omega_*(\mathbf{G})) : \lambda \in \Omega_*(\mathbf{F})\}\},
\end{aligned}$$

and from Theorem 3.5,

$$dist(\lambda, \Omega_*(\mathbf{F})) \le \alpha_*(\lambda\pi - \mathbf{F}),$$
$$dist(\lambda, \Omega_*(\mathbf{G})) \le \alpha_*(\lambda\pi - \mathbf{G}).$$

Also by Theorem 3.6, and remark 3.7(c),

$$\begin{aligned}
\alpha_*(\lambda\pi - \mathbf{F}) &= \alpha_*((\lambda\pi - \mathbf{G}) + (\mathbf{G} - \mathbf{F})) \\
&\le \alpha_*(\lambda\pi - \mathbf{G}) + w_*(\mathbf{G} - \mathbf{F}) \\
&= w_*(\mathbf{G} - \mathbf{F}), \quad \lambda \in \Omega_*(\mathbf{G}),
\end{aligned}$$

and similarly

$$\alpha_*(\lambda\pi - \mathbf{G}) \le w_*(\mathbf{F} - \mathbf{G}), \quad \lambda \in \Omega_*(\mathbf{F}).$$

Hence from these we obtain the desired result.

DEFINITION 3.9. *Let $X_0 = X - \{0\}$ and $\mathbf{f} = (f_1, \cdots, f_n)$ be an n-tuple of continuous maps from X_0 into X. We say that $\mathbf{f}$ is jointly numerically bounded if the n-tuple $\mathbf{F} = (F_1, \cdots, F_n)$ given by $F_j(x, x^*) = f_j(x)$ $(j = 1, 2, \cdots, n)$ is jointly $*$-numerically bounded, i.e.,*

$$\limsup_{r\to\infty\,\Pi_r} \frac{|<\mathbf{f}(x), x^*>|}{\|x\|\|x^*\|} < \infty.$$

In this case, the numbers $\omega_*(\mathbf{F})$, $\alpha_*(\mathbf{F})$ and the joint $*$-numerical range $\Omega_*(\mathbf{F})$ are denoted by $\omega(\mathbf{f})$, $\alpha(\mathbf{f})$ and $\Omega(\mathbf{f})$ respectively. We denote by $W^n(X)$ the vector space consisting of all numerically bounded n-tuples on X_0. Notice that $W^n(X)$ can be considered, in a natural way, as a vector space of $W_*^n(X)$, and that ω is a seminorm on $W^n(X)$. Obviously one has $Q^n(X) \subseteq W^n(X)$ and $\omega(\mathbf{f}) \le |\mathbf{f}|$

Notice that even through $\mathbf{f}$ is an n-tuple of continuous maps from X_0 into X, the joint normal component $\mathbf{f}^\nu$ of $\mathbf{f}$,

$$\mathbf{f}^\nu(x, x^*) = \left(\frac{<f_1(x), x^*>}{\|x\|\|x^*\|} x, \cdots, \frac{<f_n(x), x^*>}{\|x\|\|x^*\|} x \right),$$

is actually defined on Π_0. This is one of the reasons why we study the more general n-tuples $\mathbf{F}$ of maps from Π_0 into X.

Of course, this ambiguity disappears if X is a Banach space with a smooth unit ball. Since, in this case, there is a unique semi-inner product $[\ ,\ ]$ in X such that $[x, x] = \|x\|^2$, $x \in X$, and the formulas for $\omega(\mathbf{f})$, $\alpha(\mathbf{f})$, $\Omega(\mathbf{f})$ for a given $\mathbf{f} \in W^n(X)$, take the form

$$\omega(\mathbf{f}) = \limsup_{\|x\|\to\infty} \frac{|[\mathbf{f}(x), x]|}{\|x\|^2},$$

$$\alpha(\mathbf{f}) = \liminf_{\|x\|\to\infty} \frac{|[\mathbf{f}(x), x]|}{\|x\|^2},$$

and $\Omega(\mathbf{f}) = \bigcap_{r>0} \overline{\phi_f(E_r)}$ where $E_r = \{x \in X : \|x\| \ge r\}$ $(r > 0)$.

THEOREM 3.10. *If $\mathbf{T} = (T_1, \cdots, T_n)$ is a n-tuple of linear operators on X, then*

(a) $\Omega(\mathbf{T}) = \overline{V(\mathbf{T})}$, where $V(\mathbf{T}) = \{(f(T_1 x), \cdots, f(T_n x)) : f \in X^*, \|x\| = \|f\| = f(x) = 1\}$ denotes the joint spatial numerical range of $\mathbf{T}$.

(b) $\omega(\mathbf{T}) = \nu(\mathbf{T})$, where $\nu(\mathbf{T}) = \max\{|\lambda| : \lambda \in V(\mathbf{T})\}$ denotes the joint spatial numerical radius of $\mathbf{T}$.

Proof. The proofs follow from the definitions.

The joint $*$-asymptotic spectrum

DEFINITION 4.1. *For any $\mathbf{F} \in Q_*^n(X)$, we define*

$$d_*(\mathbf{F}) = \liminf_{r\to\infty\,\Pi_r} \frac{\|\mathbf{F}(x, x^*)\|}{\|x\|}.$$

and the joint $*$-asymptotic spectrum $\sum_*(\mathbf{F})$ of $\mathbf{F}$, as the set

$$\Sigma_*(\mathbf{F}) = \{\lambda \in \mathbf{K}^n : d_*(\lambda\pi - \mathbf{F}) = 0\},$$

where as usual π denotes the natural projection of $X \times X^*$ onto X.

For any $\mathbf{F}, \mathbf{G} \in Q_*^n(X)$ and $\mu \in K$, it is obvious that

(a) $0 \le d_*(\mathbf{F}) \le |\mathbf{F}|_*$,
(b) $d_*(\mu\mathbf{F}) = |\mu| d_*(\mathbf{F})$,
(c) $d_*(\mathbf{F} + \mathbf{G}) \le d_*(\mathbf{F}) + |\mathbf{G}|_*$,
(d) $d_*(\mathbf{F}) - |\mathbf{G}|_* \le d_*(\mathbf{F} + \mathbf{G})$,
(e) $|d_*(\mathbf{F}) - d_*(\mathbf{G})| \le |\mathbf{F} - \mathbf{G}|_*$, and
(f) $d_*(\mathbf{F}) \le |\lambda|$, $\lambda \in \sum_*(\mathbf{F})$.

THEOREM 4.2. *If $\mathbf{F}, \mathbf{G} \in Q_*^n(X)$, then*
(a) $\sum_*(\mathbf{F}) \subseteq \Omega_*(\mathbf{F})$.
(b) If $|\mathbf{F} - \mathbf{G}|_* = 0$, then $\sum_*(\mathbf{F}) = \sum_*(\mathbf{G})$.
(c) $\gamma_*(\mathbf{F}) \le |\mathbf{F}|_*$, where $\gamma_*(\mathbf{F}) = \sup\{|\lambda| : \lambda \in \sum_*(\mathbf{F})\}$ is the joint $*$-asymptotic spectral radius of $\mathbf{F}$.
(d) $\sum_*(\mathbf{F})$ is compact.

Proof. (a) It follows from the obvious inequality $\alpha_*(\mathbf{F}) \le d_*(\mathbf{F})$ and Theorem 3.6.

(b) Immediate from the above remark (e).

(c) Let $\lambda = (\lambda_1, \cdots, \lambda_n) \in \sum_*(\mathbf{F})$. By the above remark (e), we have

$$|\lambda| - |\mathbf{F}|_* \le |\lambda| - d_*(\mathbf{F}) \le d_*(\lambda\pi - \mathbf{F}) = 0.$$

(d) By the above remark (e), the mapping $\lambda = (\lambda_1, \cdots, \lambda_n) \to d_*(\lambda\pi - \mathbf{F})$ is continuous and hence $\sum_*(\mathbf{F})$ is closed. By (c), it is bounded and hence compact.

Recall that a Banach space X is said to be uniformly convex if whenever $x_n \in X$, $y_n \in X$, $\|x_n\| \le 1$, $\|y_n\| \le 1$ and $\|x_n + y_n\| \to 2$, then $\|x_n - y_n\| \to 0$.

LEMMA 4.3(CHO,1986). *The Cartesian product of finitely many uniformly convex Banach spaces can be given a uniformly convex norm.*

THEOREM 4.4. *If X is uniformly convex and $\mathbf{F} = (F_1, \cdots, F_n) \in \Omega_*^n(X)$, then $\{\lambda \in \Omega_*(\mathbf{F}) : |\lambda| = |\mathbf{F}|_*\} \subseteq \sum_*(\mathbf{F})$.*

Proof. Let $\lambda \in \Omega_*(\mathbf{F})$ and $|\lambda| = |\mathbf{F}|_*$. We may assume that $|\lambda| = |\mathbf{F}|_* = 1$. Then there exists $(x_k, x_k^*) \in \Pi_0$ such that $\|x_k\| \geq k$ and

$$\frac{< F_j(x_k, x_k^*), x_k^* >}{\|x_k\|\|x_k^*\|} \to \lambda_j \quad \text{as} \quad k \to \infty \qquad (j = 1, 2, \cdots, n).$$

Since

$$\sum_{j=1}^{n} \left| \frac{< (\lambda_j \pi + F_j)(x_k, x_k^*), x_k^* >}{2\|x_k\|\|x_k^*\|} \right|^2 \to 1 \quad \text{as} \quad k \to \infty$$

and

$$1 \geq \left(\sum_{j=1}^{n} \left\| \frac{(\lambda_j \pi + F_j)(x_k, x_k^*)}{2\|x_k\|} \right\|^2 \right)^{\frac{1}{2}},$$

$$\geq \left(\sum_{j=1}^{n} \left| < \frac{(\lambda_j \pi + F_j)(x_k, x_k^*)}{2\|x_k\|}, \frac{x_k^*}{\|x_k^*\|} > \right|^2 \right)^{\frac{1}{2}},$$

it follows that

$$\left(\sum_{j=1}^{n} \left\| \frac{(\lambda_j \pi + F_j)(x_k, x_k^*)}{\|x_k\|} \right\|^2 \right)^{\frac{1}{2}} \to 2 \quad \text{as} \quad k \to \infty.$$

So by Lemma 4.3, we have

$$\left(\sum_{j=1}^{n} \left\| \frac{(\lambda_j \pi - F_j)(x_k, x_k^*)}{\|x_k\|} \right\|^2 \right)^{\frac{1}{2}} \to 0 \quad \text{as} \quad k \to \infty.$$

Hence we obtain $d_*(\lambda\pi - \mathbf{F}) = 0$, i.e., $\lambda \in \sum_*(\mathbf{F})$.

COROLLARY 4.5. *If X is uniformly convex and $\mathbf{F} = (F_1, \cdots, F_n) \in \Omega_*^n(X)$ with $\omega_*(\mathbf{F}) = |\mathbf{F}|_*$, then $r_*(\mathbf{F}) = |\mathbf{F}|_*$.*

Proof. Since $\sum_*(\mathbf{F}) \subseteq \Omega_*(\mathbf{F})$, $r_*(\mathbf{F}) \leq \omega_*(\mathbf{F}) = |\mathbf{F}|_*$.

Also by Theorem 4.4, $\omega_*(\mathbf{F}) \leq r_*(\mathbf{F})$. Hence $r_*(\mathbf{F}) = \omega_*(\mathbf{F}) = |\mathbf{F}|_*$.

THEOREM 4.6. *Let $\mathbf{F} = (F_1, \cdots, F_n)$ be an n-tuple of maps from Π_0 into X such that*

$$\|\mathbf{F}(x, x^*)\| = \|x\| \quad \textit{for} \quad (x, x^*) \in \Pi_0.$$

Then $\lambda = (\lambda_1, \cdots, \lambda_n) \in \sum_*(\mathbf{F})$ implies $|\lambda| = 1$.

Proof. Let $\lambda \in \sum_*(\mathbf{F})$. Then by Definition 4.1, we can find $(x_k, x_k^*) \in \Pi_k$ such that

$$\|(\lambda\pi - \mathbf{F})(x_k, x_k^*)\| \leq \frac{1}{n}\|x_k\|.$$

Hence $\|\mathbf{F}(x_k, x_k^*)\| - \frac{1}{n}\|x_k\| \leq |\lambda|\|x_k\| \leq \|F(x_k, x_k^*)\|$. Using the assumption on $\mathbf{F}$ we get

$$(1 - \frac{1}{n})\|x_k\| \leq |\lambda|\|x_k\| \leq (1 + \frac{1}{n})\|x_k\|.$$

Dividing by $\|x_k\|$ and letting $n \to \infty$ completes the proof.

The joint lower $*$-numerical range

If Y is a Banach space, then Y_s^* and Y_ω^* will denote the dual of Y together with the norm(strong) and weak* topologies respectively. We denote by $J : X \to (X_s^*)_\omega^*$, the canonical isometric embedding of X into its bidual $(X_s^*)_\omega^*$. By the result of Goldstine, $J(B_R)$ is weak*-dense in B_R^{**}, where $B_R = \{x \in X : \|x\| \leq R\}$ and $B_R^{**} = \{x^{**} \in (X_s^*)_\omega^* : \|x^{**}\| \leq R\}$ $(R > 0)$.

Since our objective in this section is to study n-tuples of maps from $X_s^* \times (X_s^*)_\omega^*$ into X_s^*, we define the following sets

$\Pi_r^* = \{(x^*, x^{**}) \in X_s^* \times (X_s^*)_\omega^* : \|x^*\| = \|x^{**}\| \geq r, \|x^*\|^2 = < x^*, x^{**} >\}$ $(r > 0)$, and $\Pi_0^* = \bigcup_{r>0} \Pi_r^*$.

As before, we shall assume that Π_0^* has the norm $\times$ weak* topology induced as a subset of $X_s^* \times (X_s^*)_\omega^*$.

From Lemma 2.3, we know that each Π_r^* $(r > 0)$ and Π_0^* are connected subsets of $X_s^* \times (X_s^*)_\omega^*$. Note that each Π_r can be considered, in a natural way, as a subset of Π_r^* by means of the identification $x = J(x)$

$$\Pi_r \longleftrightarrow \Pi_r^{-1} = \{(x^*, x) : (x, x^*) \in \Pi_r\} \subseteq \Pi_r^*.$$

Thus $\mathbf{G} \in W_*^n(X^*)$ will mean that $\mathbf{G}$ is an n-tuple of continuous maps from Π_0^* into X^* such that

$$\omega_*(\mathbf{G}) = \limsup_{r \to \infty} \sup_{\Pi_r^*} \frac{| < \mathbf{G}(x^*, x^{**}), x^{**} > |}{\|x^*\|\|x^{**}\|} < \infty.$$

DEFINITION 5.1. *If $\mathbf{G} \in W_*^n(X^*)$, we define the joint lower $*$-numerical range $\Lambda\Omega_*(\mathbf{G})$ of $\mathbf{G}$ as the set*

$$\Lambda\Omega_*(\mathbf{G}) = \bigcap_{r>0} \overline{\Psi_{\mathbf{G}}(\Pi_r^{-1})}$$

where

$$\Psi_{\mathbf{G}}(x^*, x^{**}) = \frac{< \mathbf{G}(x^*, x^{**}), x^{**} >}{\|x^*\|\|x^{**}\|}, \quad (x^*, x^{**}) \in \Pi_0^*.$$

and

$$\Psi_G(x^*, Jx) = \frac{< G(x^*, Jx), Jx >}{\|x^*\|\|J_x\|}, = \frac{< x, G(x^*, Jx) >}{\|x\|\|x^*\|} \quad (x, x^*) \in \Pi_0.$$

It is clear that $\Lambda\Omega_*(G) \subseteq \Omega_*(G)$, *where, as usual,* $\Omega_*(G)$ *denotes the joint* $*$*-numerical range of* G.

LEMMA 5.2(CANAVATI, 1979). Π_r^{-1} *is norm* $\times$ *weak* dense in* $\Pi_r^*(r > 0)$, *i.e., in the topology of* $X_s^* \times (X_s^*)_\omega^*$.

THEOREM 5.3. *If* $G \in W_*^n(X^*)$, *then* $\Lambda\Omega_*(G) = \Omega_*(G)$.

Proof. We have only to prove that $\Omega_*(G) \subseteq \Lambda\Omega_*(G)$. Since $\Psi_G : \Pi_0^* \to K^n$ is a continuous map, we have

$$\Psi_G(\overline{\Pi_r^{-1}}) \subseteq \overline{\Psi_G(\Pi_r^{-1})} \quad (r > 0).$$

From the previous lemma,

$$\Psi_G(\Pi_r^*) \subseteq \overline{\Psi_G(\Pi_r^{-1})} \quad (r > 0).$$

Hence $\Omega_*(G) = \bigcap_{r>0} \overline{\Psi_G(\Pi_r^*)} \subseteq \bigcap_{r>0} \overline{\Psi_G(\Pi_r^{-1})} = \Lambda\Omega_*(G)$.

The numerical range for vector fields on the unit space

Let X be a Banach space and $S = \{x \in X : \|x\| = 1\}$ be the unit sphere in X. Let $\Phi = (\Phi_1, \cdots, \Phi_n)$ be an n-tuple of continuous maps from S into X, i.e., an n-tuple of vector fields on S. We say that $\tilde{\Phi}(x) = \|x\|(\Phi_1(\|x\|^{-1}x), \cdots, \Phi_n(\|x\|^{-1}x))$, $x \neq 0$ is numerically bounded. In this case, we let $\omega(\Phi) = \omega(\tilde{\Phi}), \alpha(\Phi) = \alpha(\tilde{\Phi})$ and $\Omega(\Phi) = \Omega(\tilde{\Phi})$.

If we set $\Pi = \{(u, u^*) \in X \times X^* : \|u\| = \|u^*\| = < u, u^* >= 1\}$, then Π is a connected subset of $X \times X^*$ with the norm $\times$ weak* topology.

THEOREM 6.1. *Let* $\Phi = (\Phi_1, \cdots, \Phi_n)$ *be a numerically bounded n-tuple of vector fields on* S. *Then*

(a) $\omega(\Phi) = \sup_\Pi | < \Phi(u), u^* > |$

(b) $\alpha(\Phi) = \inf_\Pi | < \Phi(u), u^* > |$

(c) $\Omega(\Phi) = \{< \Phi(u), u^* >: (u, u^*) \in \Pi\}^-$, *where* $\overline{E}$ *denotes the closure of* E.

Proof. (a) and (b) follow from

$$\frac{< \tilde{\Phi}(x), x^* >}{\|x\|\|x^*\|} = \frac{< \|x\|\Phi(\|x\|^{-1}x), x^* >}{\|x\|\|x^*\|} =< \Phi(\|x\|^{-1}x), \|x^*\|^{-1}x^* > =< \Phi(u), u^* >,$$

where $u = \|x\|^{-1}x$, $u^* = \|x^*\|^{-1}x^*$ and $(u, u^*) \in \Pi$.

Now (c) becomes evident.

From this last result we see that $\Omega(\Phi)$ coincides with the closure $\overline{V(\Phi)}$ of the numerical range $V(\Phi)$ of an n-tuple $\Phi = (\Phi_1, \cdots, \Phi_n)$ of continuous maps from S into X.

Acknowledgement. The author would like to thank the KOSEF for his partially financial support, 1991 (911-0102-026-1).

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〈국문초록〉

非線型 函數族에 관한 스펙트럼 理論

Banach 空間에서 結合的 *-準有界(*-quasibounded) 函數空間 및 결합적 *-數値的 有界(*-numerically bounded)인 連續函數族의 함수공간의 特性을 밝히고, 이 결합적 *-수치적 유계인 함수족의 結合數域(numerical range)을 정의하여 이의 여러가지 性質을 조사한다. 특히, 이 數域은 C^n의 compact 連結 部分集合이고, 이 함수족이 線形 作用素族일 때는 이 空間的 수역의 閉包와 일치한다. 또한 결합적 *-준유계인 함수족의 結合的 點近似 스펙트럼을 정의하여 이 스펙트럼이 수역의 compact 부분집합임을 보이고, 一樣볼록 공간에서 이 스펙트럼과 수역과의 관계를 밝힌다. 그리고 Banach 공간의 雙對空間에서 연속함수족의 아래 *-수역을 정의하여 이 수역이 結合數域과 일치하기 위한 條件을 구한다.